Abstract submitted to 16th IWWWFB, Hiroshima, Japan, 22–25 April 2001

Non-axisymmetric trapping structures in the three-dimensional water-wave problem

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Introduction

Trapped modes are free oscillations with finite energy of an unbounded fluid for which the fluid motion is essentially confined to the vicinity of a fixed structure. In recent years it has been discovered that such modes exist in the three-dimensional linearized water-wave problem and may be supported at specific frequencies by certain 'trapping structures' [1,2]. The existence of a trapped mode at a particular frequency implies the non-uniqueness, or even non-existence, of the solution to the scattering or radiation problem at that frequency.

Axisymmetric trapped modes in the presence of axisymmetric structures may be constructed by an inverse procedure in which the main idea is to specify an axisymmetric velocity field that decays at large distances, and then to seek stream surfaces that correspond to rigid structures [1]. A time-harmonic circular ring source of radius c, and with a vertical axis of symmetry, is placed in the free surface. There is no wave propagation to infinity at the frequencies given by $Kc = j_{0,n}$, where $K = \omega^2/g$, ω is the radian frequency, g is the acceleration due to gravity, and $j_{0,n}$ is the nth zero of the Bessel function J_0. Axisymmetric stream surfaces of this flow correspond to particular toroidal structures intersecting the free surface which, by construction, are able to support free oscillations of the fluid; an example of such a structure is shown in figure 1. Subsequently, the construction was extended [2] to give non-axisymmetric trapped modes in the presence of axisymmetric toroidal structures by allowing the strength of the ring source, and hence the corresponding velocity field, to have a sinusoidal azimuthal variation. A hydrodynamic analysis has been performed for axisymmetric trapping structures [3] and singular behaviour of, for example, the added mass and damping is observed in the vicinity of the trapped-mode frequency.

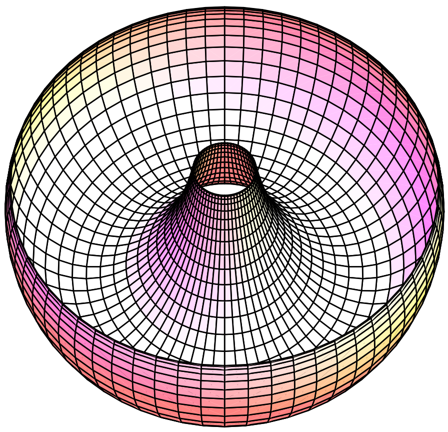

Figure 1: Perspective view of the submerged surface of an axisymmetric trapping structure.

The question arises 'Can non-axisymmetric trapping structures be found?'. The present work answers this question in the affirmative and, in particular, it is demonstrated that *non-axisymmetric* trapping structures can be constructed from an *axisymmetric* velocity field.

Construction of trapped modes

It is most convenient to work in terms of toroidal coordinates (r, θ, β) with $r > 0$, $0 \leq \theta < \pi$, and $0 \leq \beta < 2\pi$ [4], which are related to rectangular Cartesian coordinates (x, y, z) by

$$x = (c - r\cos\theta)\cos\beta, \quad y = (c - r\cos\theta)\sin\beta, \quad z = r\sin\theta, \tag{1}$$

where c is the radius of a circular ring in the free surface (perhaps coinciding with a ring source as described above), and z is directed vertically downwards with $z = 0$ corresponding to the mean free surface. Thus, β is an azimuthal angle measured around the z axis and, in any vertical plane through the z axis, (r, θ) are equivalent to plane polar coordinates with origin at $R = (x^2+y^2)^{1/2} = c$, $z = 0$.

As in previous work [1,2], a flow field is first specified in terms of a velocity potential

$$\Phi(r, \theta, \beta, t) = \phi(r, \theta, \beta) \cos \omega t, \tag{2}$$

where t is time and ω is the radian frequency of the fluid oscillations. Let $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\beta$ be unit vectors in the r, θ and β directions respectively. With the time dependence removed, the velocity is

$$\nabla\phi = \frac{\partial\phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\mathbf{e}_\theta + \frac{1}{c - r\cos\theta}\frac{\partial\phi}{\partial\beta}\mathbf{e}_\beta \tag{3}$$

and it is required to determined a surface $r = r(\theta, \beta)$ such that

$$\nabla\phi.\boldsymbol{n} = 0, \tag{4}$$

for all normals

$$\boldsymbol{n} = -\mathbf{e}_r + \frac{1}{r}\frac{\partial r}{\partial\theta}\mathbf{e}_\theta + \frac{1}{c - r\cos\theta}\frac{\partial r}{\partial\beta}\mathbf{e}_\beta \tag{5}$$

to the surface. In other words, the kinematic condition to be satisfied on a structural surface is

$$q_\theta\frac{\partial r}{\partial\theta} + q_\beta\frac{\partial r}{\partial\beta} = q_r \tag{6}$$

where

$$q_r = \frac{\partial\phi}{\partial r}, \quad q_\theta = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta}, \quad q_\beta = \frac{1}{(c - r\cos\theta)^2}\frac{\partial\phi}{\partial\beta}. \tag{7}$$

Now for $q_\theta \neq 0$ equation (6) can be rewritten as

$$\frac{\partial r}{\partial\theta} + \frac{q_\beta}{q_\theta}\frac{\partial r}{\partial\beta} = \frac{q_r}{q_\theta} \tag{8}$$

so that

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{q_r}{q_\theta} \tag{9}$$

on the curves

$$\frac{\mathrm{d}\beta}{\mathrm{d}\theta} = \frac{q_\beta}{q_\theta}. \tag{10}$$

The last two equations determine the so-called characteristic curves [5, Chapter II, §1]. Under certain not very restrictive conditions, given an initial curve Γ in the free surface defined by $r = r(0, \beta)$, equations (9)–(10) can be integrated from initial points $(r(0, \beta_0), 0, \beta_0)$ on Γ to determine curves that are everywhere parallel to the velocity field. Thus, given the velocity field and an appropriately chosen initial closed curve Γ in the free surface, a stream surface can be generated by simultaneous integration over $0 \leq \theta \leq \pi$ of the two first-order differential equations (9)–(10). For a velocity field generated from a ring source that is singular in the free surface at $R = c$, a sensible choice for Γ is a closed curve surrounding the origin and entirely within $R = c$.

For the special case in which the specified flow is axisymmetric, so that $q_\beta = 0$, equation (10) gives immediately that β is constant on any characteristic curve and it is sufficient to integrate (9) to determine the characteristics. Although the velocity field is axisymmetric there is no requirement that the initial curve Γ must also be axisymmetric. Thus, non-axisymmetric stream surfaces can be generated from axisymmetric velocity fields!

Stream-function approach

Another approach to the axisymmetric flow case is to express the given velocity field in terms of a function $\psi(r,\theta,\beta)$ so that

$$\nabla\phi = -\frac{1}{r(c - r\cos\theta)}\frac{\partial\psi}{\partial\theta}\mathbf{e}_r + \frac{1}{r(c - r\cos\theta)}\frac{\partial\psi}{\partial r}\mathbf{e}_\theta; \tag{11}$$

the particular form arises from the requirement that $\nabla^2\phi = 0$. For some constant C, a normal to a surface S defined by

$$\psi(r,\theta,\beta) = C \tag{12}$$

is

$$\nabla\psi = \frac{\partial\psi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{e}_\theta + \frac{1}{c - r\cos\theta}\frac{\partial\psi}{\partial\beta}\mathbf{e}_\beta \tag{13}$$

and the construction ensures that $\nabla\psi$ is perpendicular to $\nabla\phi$ everywhere on S, and hence S is a stream surface of the flow. Given an axisymmetric velocity field, (11) can be solved to determine ψ; this is probably best done in a different coordinate system. In most applications ψ is chosen to be axisymmetric (and often called the Stokes' stream function) so that ψ is independent of β and the surfaces S are also axisymmetric. This is the approach initially adopted in this problem in order to generate axisymmetric trapping structures from axisymmetric velocity fields [1]. However, if

$$\psi(r,\theta,\beta) = \Psi(r,\theta) \tag{14}$$

is a particular solution to (11) then

$$\psi(r,\theta,\beta) = \Psi(r,\theta) + \chi(\beta) \tag{15}$$

is also a solution for any reasonable $\chi(\beta)$. In general, the surfaces S generated from (12) using (15) are not axisymmetric.

An example of a non-axisymmetric trapping structure

Many types of non-axisymmetric trapping structures can be generated in the manner described above. For instance, the structure shown in figure 1 can be distorted so that the inner and outer radii are smoothly varying functions of the azimuthal angle β. Here we present another example in which sections of different radius are joined together.

The axisymmetric flow is specified in terms of the potential for a ring source of radius one. The geometry of the structure shown in figure 2 is defined explicitly, in a manner described elsewhere [6]. Three 'patches' are defined in one quadrant as follows. Patch 1 consists of a partial torus with inner waterline radius 0.2 restricted to the range $\beta \in (0, \pi/4)$, patch 2 consists of a partial torus with inner waterline radius 0.3 restricted to the range $\beta \in (\pi/4, \pi/2)$, and patch 3 is the portion of the azimuthal plane $\beta = \pi/4$ between the generating sections of the first two patches (planes of constant β are also stream surfaces of an axisymmetric flow). After reflection about the planes $x = 0$ and $y = 0$ a *non-axisymmetric* closed structure is formed, with the property that its surface coincides with the *axisymmetric* stream surfaces generated by the ring source. The toroidal radial coordinates of the generating sections for the first two patches are defined by economized polynomials of degree 10 in the angle θ. The maximum error in these polynomial approximations is 5×10^{-6}.

Figure 3 shows the heave added-mass coefficient for this structure, computed by the program WAMIT with quadratic B-spline representation of the potential and exact representation of the geometry as defined by the above polynomial approximations. Three different results are shown, with N=48, 108, and 300 unknowns in the linear system of equations, corresponding to subdivisions

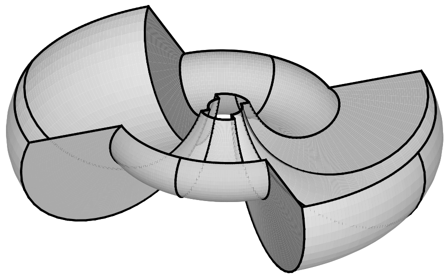

Figure 2: Perspective view of a non-axisymmetric trapping structure. The dark lines show the boundaries of each patch and its reflections about the planes of symmetry.

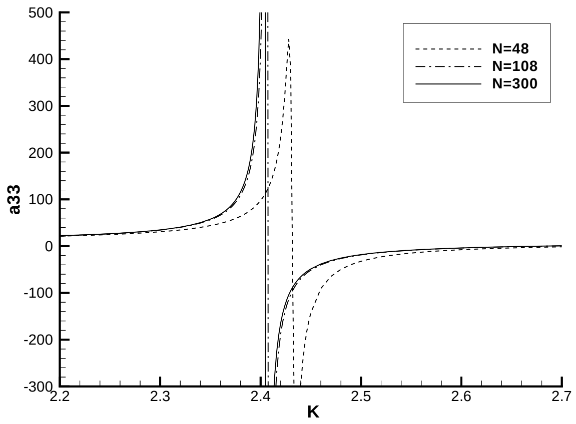

Figure 3: Heave added-mass coefficient a_{33}, normalized with respect to the radius of the ring source and fluid density vs. the wavenumber K.

N	panels	K_0	$\lvert a_{33}\rvert$
48	2×2	2.430	374
108	4×4	2.406	4,440
300	8×8	2.4048	69,000

Table 1: Singular wave number K_0 and added mass a_{33} as a function of the number of unknowns N.

of each patch into 2×2, 4×4, and 8×8 elements. These results are computed in the range shown using 102 closely spaced wavenumbers. In the vicinity of the singular wavenumber $K = j_{0,1} \approx 2.4048$ the increment is $\Delta K = 0.0001$. The value K_0 of the wavenumber where the added-mass coefficient changes sign and the maximum value of this coefficient are shown in table 1. These numerical results give strong supporting evidence for the existence of non-axisymmetric trapping structures.

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