Trapping of waves by a submerged elliptical torus

by M. McIver* & R. Porter†

* Department of Mathematical Sciences, Loughborough University, Loughborough, Leicestershire, LE11 3TU, UK

† School of Mathematics, University of Bristol, University Walk, Bristol, BS8 1TW, UK

Introduction

In linear water-wave theory a trapped mode is defined to be a non-zero solution of the homogeneous equations and boundary conditions, which doesn't radiate waves and has finite energy. The existence of such modes supported by three-dimensional, surface-piercing bodies in the open sea was first established by McIver & McIver[1]. They constructed an axisymmetric potential and defined the shape of the body to be one of the stream surfaces of the flow. In practice, however, it would be more useful to be able to specify the shape of the obstacle and determine whether there are any trapped modes for that geometry. The purpose of this work is to show how this may be done for certain classes of bodies by exploiting the link between three-dimensional trapped modes and zeros of transmission for two-dimensional obstacles. First a plane wave approximation to a trapped mode is derived, then a numerical technique for finding exact trapped mode wave numbers is described. Finally some results which show that modes may be trapped by a submerged elliptical torus are presented.

The plane wave approximation

A two-dimensional body which is totally submerged in deep water and which possesses a zero of transmission is rotated through 360^o about a vertical axis outside the body, to form a submerged torus. The axis of rotation is at a distance c from a reference axis through the body where $c/a \gg 1$ and a is a typical dimension of the original two-dimensional body. Rectangular Cartesian coordinates are chosen so that the origin is in the mean free surface above the centre of the torus and the z-axis points vertically upwards. An approximation to the trapped mode potential is sought which represents a standing wave in the region $r = \sqrt{x^2 + y^2} \ll c$ and is zero for $r \gg c$. The two potentials are then matched across the torus using the plane wave approximation of Simon[2]. The axisymmetric standing wave is given by

$$Re[\phi \, \mathrm{e}^{-\mathrm{i}\omega t}] = Re[J_0(kr) \, \mathrm{e}^{kz - \mathrm{i}\omega t}] \tag{1}$$

where J_0 is the zero-order Bessel function of the first kind, ω is the angular frequency of oscillation, $k = \omega^2/g$ and g is the acceleration due to gravity. By writing $kr = k(r-c) + kc$ and using the addition theorem and the large argument expansion for Bessel functions, it may be shown that

$$\phi \sim \sqrt{\frac{1}{2\pi kc}} \, \mathrm{e}^{\mathrm{i}(kc - \pi/4) + kz} \left[\mathrm{e}^{\mathrm{i}k(r-c)} + \mathrm{i} \, \mathrm{e}^{-2\mathrm{i}kc} \, \mathrm{e}^{-\mathrm{i}k(r-c)} \right], \quad kc \gg 1, \quad k(r-c) = O(1). \tag{2}$$

Near the body of the torus this represents a plane wave which propagates outwards along a radial line and which is totally reflected by the two-dimensional body formed by the cross-section of the torus. Clearly this can only occur if the wave number corresponds to a point of zero transmission of the two-dimensional body and the reflection coefficient is given by

$$R = \mathrm{i} \, \mathrm{e}^{-2\mathrm{i}kc} \, . \tag{3}$$

This last equation gives possible values of kc in terms of the phase of the reflection coefficient. So, if $R = \mathrm{e}^{\mathrm{i}\delta}$ is known from the solution of the two-dimensional scattering problem, then

$$kc = \frac{\pi}{4} - \frac{\delta}{2} + n\pi, \tag{4}$$

where n is an arbitrary integer. (Newman[3] used matched asymptotic expansions to perform a similar but more rigorous analysis of a floating torus at small but non-zero values of R.) It is interesting to observe that the corresponding result for a wide-spacing approximation to two-dimensional modes trapped between two bodies is $kc = -\delta/2 + n\pi$, and so there is a difference of $\pi/4$ between the two and three dimensional approximate values of kc.

Numerical location of trapped modes

The analysis of the previous section is only approximate and so it is not obvious whether the torus can support trapped modes or whether it just has highly tuned resonances. In this section a method is described which both demonstrates that there is a perturbation of the torus which supports a trapped mode, and allows its accurate computation.

An exact axisymmetric trapped mode for a submerged torus is sought where, without loss of generality, the potential ϕ is assumed to be real. An application of Green's theorem to ϕ and the ring source gives ϕ at an arbitrary point in the fluid in terms of an integral of ϕ multiplied by the normal derivative of the ring source around the boundary of the torus. This is then differentiated, evaluated on the boundary of the torus and integrated by parts to give

$$\int_0^{2\pi} H(r(t), z(t); r(\tau), z(\tau))\, q(t)\, \mathrm{d}t = \psi_c, \quad 0 \le \tau < 2\pi, \tag{5}$$

where $q(t) = \partial\phi/\partial s\, \mathrm{d}s/\mathrm{d}t$. The function $\partial\phi/\partial s$ is the tangential derivative of ϕ on the boundary of the cross-section of the torus, $(r(t), z(t))$ are the radial and vertical coordinates of a point on the torus, ψ_c is the unknown, real, constant value of the Stokes' stream function on the torus and

$$\begin{aligned} H(r, z; r_0, z_0) &= \frac{2krkr_0}{\pi} \int_0^\infty \frac{(\nu \sin \nu kz - \cos \nu kz)(\nu \sin \nu kz_0 - \cos \nu kz_0) I_1(\nu kr_<) K_1(\nu kr_>)}{\nu^2 + 1}\, \mathrm{d}\nu \\ &- \mathrm{i}\pi krkr_0\, \mathrm{e}^{k(z+z_0)}\, J_1(kr_<) H_1(kr_>), \end{aligned} \tag{6}$$

where J_1, H_1, I_1 and K_1 are the usual Bessel and modified Bessel functions, $r_< = \min(r_0, r)$ and $r_> = \max(r_0, r)$. In addition, to ensure zero circulation around the torus

$$\int_0^{2\pi} q(t)\, \mathrm{d}t = 0. \tag{7}$$

The boundary of the torus is assumed to be smooth and so $q(t)$ is approximated by a series of trigonometric functions

$$q(t) \approx \sum_{n=1}^{2N} a_n u_n(t), \quad \text{where} \quad u_{2n-1}(t) = \sin nt, \quad u_{2n}(t) = \cos nt. \tag{8}$$

(The omission of the constant term ensures that (7) is automatically satisfied.) An application of Galerkin's method yields a real homogeneous matrix equation for the coefficients $\{a_n\}$

$$\sum_{n=1}^{2N} K_{mn} a_n = 0, \quad \text{where} \quad K_{mn} = \int_0^{2\pi} \int_0^{2\pi} \mathrm{Re}[H(r(t), z(t); r(\tau), z(\tau)] u_m(\tau) u_n(t)\, \mathrm{d}\tau\, \mathrm{d}t, \quad m = 1, \ldots, 2N. \tag{9}$$

In addition, in order to satisfy the imaginary part of (5)

$$S \equiv \sum_{n=1}^{2N} f_n a_n = 0, \quad \text{where} \quad f_n = \int_0^{2\pi} kr(t)\, \mathrm{e}^{kz(t)}\, J_1(kr(t)) u_n(t)\, \mathrm{d}t. \tag{10}$$

Thus, a trapped mode wave number is a value of k for which $\det(K_{mn}) = 0$ and the resulting eigenfunction satisfies the side condition (10).

Results and discussion

Numerical calculations were performed for a submerged elliptical torus whose boundary is parameterised by

$$(r, z) = (c + a\cos t, -d + b\sin t), \quad 0 \le t < 2\pi. \tag{11}$$

Computations for a submerged ellipse with $b/a = 0.16$ and $d/a = 0.25$ show that a zero of transmission exists at $ka = 0.525$ and the plane wave approximation predicts that a torus with $c/a = 3.600$ supports a trapped mode. Exact trapped modes are sought for an elliptical torus with the same values of b/a and d/a and neighbouring values of c/a and ka. Figure 1 illustrates the curves $\det(K_{mn}) = 0$ and $\tilde{S} \equiv \sum_{n=1}^{2N} f_n \tilde{a}_n = 0$ as functions of c/a and ka in the vicinity of $c/a = 3.600$ and $ka = 0.525$, where the coefficients $\{\tilde{a}_n\}$ are the coefficients in the eigenfunction for the eigenvalue of K_{mn} which has the smallest magnitude. At the point where the two curves cross $\tilde{S} = S$ and so

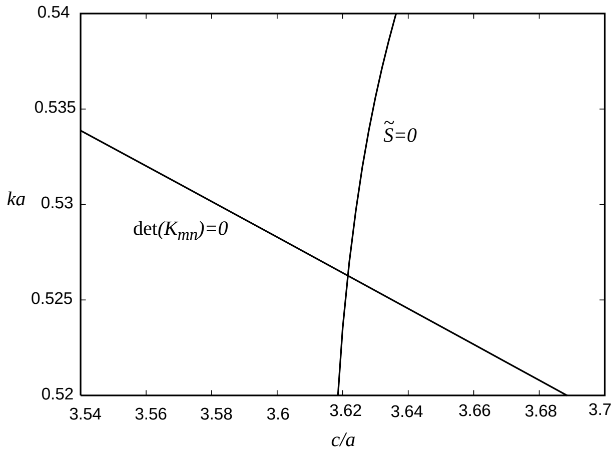

Figure 1: Crossing of the curves $\det(K_{mn}) = 0$ and $\tilde{S} = 0$ for an elliptical torus, $b/a = 0.16$, $d/a = 0.25$.

$\det(K_{mn}) = 0$ and $S = 0$ there. The two curves are generated by determining where real quantities change sign and the fact that the curves cross rather than touch gives confidence that a trapped mode exists, even though the values of ka and c/a may not be found exactly. Table 1 summarises the trapped mode parameters obtained by this method for different values of b/a and d/a.

	b/a					
d/a	0.04	0.08	0.12	0.16	0.2	0.24
0.3	-	-	-	-	4.22430 0.48159	3.98209 0.45113
0.25	-	3.96860 0.56043	3.71152 0.56145	3.62155 0.52627	3.66547 0.45148	4.21728 0.28664
0.2	3.25850 0.65717	3.26333 0.61093	3.33559 0.53944	3.57864 0.42465	-	-
0.15	3.03694 0.63019	3.23079 0.52534	3.75368 0.36853	-	-	-
0.1	3.31776 0.47983	4.43568 0.27962	-	-	-	-

Table 1: Summary of values of c/a (upper entry) and ka (lower entry) for trapping by an elliptical torus for selected values of aspect ratio, b/a, and submergence, d/a.

Figures 2 and 3 illustrate the non-dimensional heave added mass and damping for a torus with $b/a = 0.16$, $d/a = 0.25$ and $c/a = 3.62155$, calculated using WAMIT with a varying number of panels. The panels are distributed at equal azimuthal angles and at equal values of the arc length around the boundary of the torus. It is clear from the figures there is a large spike in the added mass and damping at $ka \approx 0.527$ which supports the existence of a trapped mode at that value.

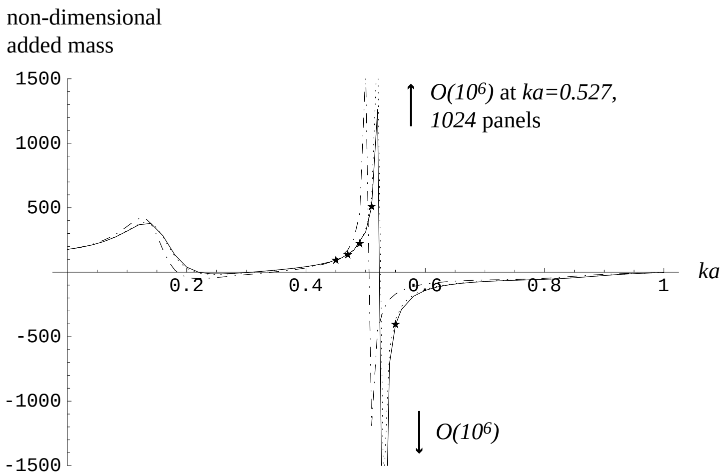

Figure 2: Heave added mass for a submerged elliptical torus, $b/a = 0.16$, $d/a = 0.25$, $c/a = 3.62155$. Number of panels on 1/4 torus, $\star$: 1920, —: 1024, $\cdots$: 512, $-\cdot-$:256

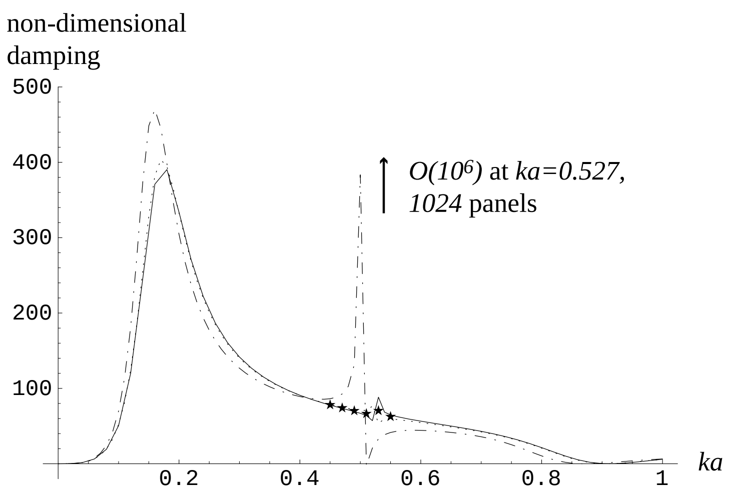

Figure 3: Heave damping for a submerged elliptical torus, $b/a = 0.16$, $d/a = 0.25$, $c/a = 3.62155$. Number of panels on 1/4 torus, $\star$: 1920, —: 1024, $\cdots$: 512, $-\cdot-$:256

References

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