Wave interactions in the coastal zone

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1. Introduction. In the coastal zone free-surface waves are often quite non-linear, due to shoaling from deeper water where they have been generated by wind or other means. To understand and predict the further evolution in the coastal zone wave-interaction processes are studied by several means. The results of such studies may be applicable to spectral wave models like WAM (*e.g.* see Komen *et al.*, 1994) and SWAN (Booij *et al.*, 1999), as well as to the drift motion of ships moored in shallow water.

2. Cumulant-closure approaches. In the coastal zone we usually have water of restricted depth, for which case Boussinesq-like equations form an adequate description, *e.g.* Dingemans (1997). For simplicity of the discussion we now treat only 1D horizontal wave propagation in this section. Examples of this approach have been given, amongst others, by Rasmussen (1998) and by Becq-Girard *et al.* (1999). Also approaches based on the Laplace equations are followed, *e.g.* Eldeberky and Madsen (1999). See also Dingemans (2000) for a discussion on the above models. Starting with the Boussinesq-like equations given by Madsen and Sørensen (1993) for a slowly-varying bottom formulated in the free-surface elevation $\zeta(x,t)$ and the vertically integrated velocity $q = \int dz\, u(x,z,t)$. The Fourier expansions of ζ and q are written as:

$$(2.1)\qquad \begin{pmatrix} \zeta(x,t) \\ q(x,t) \end{pmatrix} = \sum_{m=-\infty}^{\infty} \begin{pmatrix} 1 \\ \dfrac{\omega_m}{k_m(x)} \end{pmatrix} A_m(\beta x) \exp\left[i\left(\omega_m t - \psi_m(x)\right)\right]$$

where $k_m = \partial\psi_m/\partial x$ and $\beta \ll 1$ denotes the slow variation of the amplitudes. Inserting these series in the Boussinesq-like equations and keeping only the lowest-order terms in β leads to an amplitude equation of the following form

$$(2.2)\qquad \frac{dA'_q}{dx} = \left(\sigma_q \frac{dh}{dx} - ik_q(x)\right) A'_q + i \sum_{m=-\infty}^{\infty} J_{m,q-m} A'_m A'_{q-m} \ ,$$

where $A'_q = A_q \exp\left[-i\psi_q(x)\right]$ and the linear shoaling coefficient σ_q and the interaction coefficient $J_{m,q-m}$ are long expressions, independent of the bottom slope. Notice that reflection has been neglected.

To obtain a model in terms of the (discrete) variance (power spectrum) $E_q = \left\langle A'_q A'^*_q \right\rangle$ the evolution equation (2.2) is multiplied with A'^*_q; the conjugate evolution equation is multiplied with $A\prime_q$ and the averaged. Both contributions are added and of the results the ensemble average is taken. The result is an evolution equation for the discrete spectral values:

$$(2.3)\qquad \frac{dE_q}{dx} = 2\sigma_q \frac{dh}{dx} E_q - 2 \sum_{-\infty}^{\infty} J_{m,q-m}\, \mathrm{Im}\left\{B_{m,q-m}\right\} \ ,$$

where the bi-spectrum is defined by $B_{m,q-m} = \left\langle A'_m A'_{q-m} A'^*_q \right\rangle$.

In the same way an evolution equation for the bi-spectrum can be derived (*e.g*, see Rasmussen, 1998):

$$(2.4)\qquad \begin{aligned} \frac{dB_{m,q-m}}{dx} &= \left[\left(\sigma_m + \sigma_{q-m} + \sigma_q\right)\frac{dh}{dx} - i\delta k(x)\right] B_{m,q-m} \\ &\quad + i \sum_{-\infty}^{\infty} \left(J_{n,m-n} T_{n,m-n,q-m,-q} + J_{n,q-m-n} T_{n,q-m-n,m,-q} - J_{n,q-n} T_{-n,n-q,m,q-m}\right) \ , \end{aligned}$$

where $\delta k(x) = k_m(x) + k_{q-m}(x) - k_q(x)$ is the wave-number mismatch, and $T_{n,m-n,q-m,-q}$ is the discrete tri-spectrum defined as $T_{n,m-n,q-m,-q} = \left\langle A'_n A'_{m-n} A'_{q-m} A'^*_q \right\rangle$.

3. Closure hypotheses. Some closure is needed now because the tri-spectrum is in principle unknown. What is usually done is to assume the wave field to be Gaussian, which allows one to discard the fourth-order cumulant. An n-th order moment of random quantities $a_1 \cdots a_n$ can be reduced to a sum of products of lower-order moments, plus a irreducible term, the n-th order cumulant:

$$(3.1)\qquad \left\langle a_1 \cdots a_n \right\rangle = \sum_{j=1}^{n-1} \left\langle a_1 \cdots a_j \right\rangle \left\langle a_{j+1} \cdots a_n \right\rangle + \left\langle a_1 \cdots a_n \right\rangle^C \ .$$

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For a discussion of cumulants[1] for both random variables and random fields is referred to Monin and Yaglom (1975, pp. 223 ff.) or to Kendall and Stuart (1977, Chapter 3).

When also near-stationarity is assumed and when shallow-water wave approximations are used to simplify the coefficients, the following evolution equation for the bi-spectrum results (*i.e.* see Rasmussen[2], 1998, Eqs. (9.20) and (9.21)):

$$\frac{dE_q}{dx} = 2\left(\sigma_q \frac{dh}{dx}\right) E_q - 2 \sum_{m=-\infty}^{\infty} J_{m,q-m} \operatorname{Im}\left\{B_{m,q-m}\right\} , \tag{3.2a}$$

and

$$\begin{aligned} \frac{dB_{m,q-m}}{dx} &= \left[(\sigma_m + \sigma_{q-m} + \sigma_q) \frac{dh}{dx} - i\delta k(x)\right] B_{m,q-m} \\ &\quad + 2iJ_{m,q-m}\left(\frac{k_m}{k_q} E_q E_{q-m} + \frac{k_{q-m}}{k_q} E_q E_m - \frac{k_q}{k_m} E_m E_{q-m}\right) . \end{aligned} \tag{3.2b}$$

For use in wave models like WAM and SWAN it is advantageous to decouple these equations.However, this can only be done in a very approximate way and the resulting spectral energy evolution equation gives no evolution in case of a horizontal bottom. The fourth-order cumulant discard hypothesis has been used extensively, see, *e.g.* Rasmussen (1998) and Eldeberky and Madsen (1999).

In turbulence research it is well known that the fourth-order cumulant discard hypothesis (also known as the Millionschikov hypothesis, see, *e.g.*, Monin and Yaglom, 1975, p. 241 and §19.3). Applying this hypothesis leads to an evolution equation for the energy which leads to negative energies eventually because of the viscous damping. In water waves a similar damping is present because usually a cut-off frequency is used and increasingly shorter waves come in the region beyond the cut-off frequency and lead to a form of damping. The inconsistency in using the Gaussianity assumption can be made clear in the following way. The energy density evolution equation shows the generation of bound waves. With the presence of bound and free waves in the wave field, it cannot be true anymore that the totality of the wave field remains Gaussian (or, otherwise stated, uncoupled and therefore linear). Janssen (1991) shows that it is a necessity for having transfer of energy over the various components of the spectrum to have the fourth-order cumulant to be different from zero. He therefore takes the sixth-order cumulant to be zero.

A different, but related view has been given by Holloway (1980). Instead of taking the n-th order cumulant to be zero, $\langle a_1 \cdots a_n \rangle^C = 0$, Holloway (1980) substitutes for the n-th order cumulant a term linear in the $(n-1)$-th cumulant $\langle a_1 \cdots a_{n-1} \rangle^C$ with a fore-factor which is an unknown function of lower-order cumulants. Effectively what is happening is that instead of supposing the fourth-order cumulant to be zero, the difference between the fourth-order cumulant and some linear functional of the triple correlation is supposed to be zero. Following Holloway (1980), the kinetic equation for the spectral action density N_ℓ can be written in the following general form:

$$\frac{\partial N_\ell}{\partial t} = \int_\Delta dk_m dk_n \Gamma_{\ell mn} \left(N_m - N_\ell\right) N_n \theta_{\ell nm} , \tag{3.3}$$

with $\int_\Delta$ denoting the integration over the wave numbers satisfying $\boldsymbol{k}_\ell + \boldsymbol{k}_m + \boldsymbol{k}_n = 0$, the $\Gamma_{\ell mn}$ are the interaction coefficients depending on the specific model and $\theta_{\ell mn} = \operatorname{Re}\left\{\left[\mu_{\ell mn} + i\left(\omega_{\ell mn}\right)\right]^{-1}\right\}$. The coupling coefficient thus appears as an frequency uncertainty coefficient among three interaction waves, indicating a broadening of the resonance condition. For the determination of $\mu_{\ell mn}$ Holloway suggests to identify it with the sum of the individual interaction rates $\mu_{\ell mn} = \eta_\ell + \eta_m + \eta_n$ with the fundamental interaction rate η_ℓ given by $\eta_\ell = \int_\Delta dk_m dk_n \Gamma_{\ell mn} N_n \theta_{\ell mn}$. Equation (3.3) together with the relations for $\theta_{\ell mn}$, $\mu_{\ell mn}$ and η_ℓ constitute a closed set of equations describing the evolution of strongly-interacting waves. In the approach of Becq-Girard em et al. (1999) an approximate version of the approach of Holloway has been followed. Here the parameter $\mu_{\ell mn}$ has been replaced by a fixed parameter K to be chosen beforehand. No simple prescription how to choose K is available.

4. Perturbation-series approaches. Another approach to describe the non-linear triad interactions is by the direct application of Stokes' second-order wave theory to directional random waves. This has been done in a classical Stokes' second-order perturbation-series approach for deep-water waves by *e.g.* Masuda *et al.* (1979), and for an arbitrary water depth with a horizontal bed by *e.g.* Dean and Sharma (1981) and Laing (1986). Willebrand (1975) started from a variational principle and derived similar results which are also applicable to mildly sloping sea beds.

[1]In the Russian literature *cumulant* is also termed *semi-invariant.*

[2]Rasmussen also considers effects of wave dissipation due to wave breaking, which effects we here ignore.

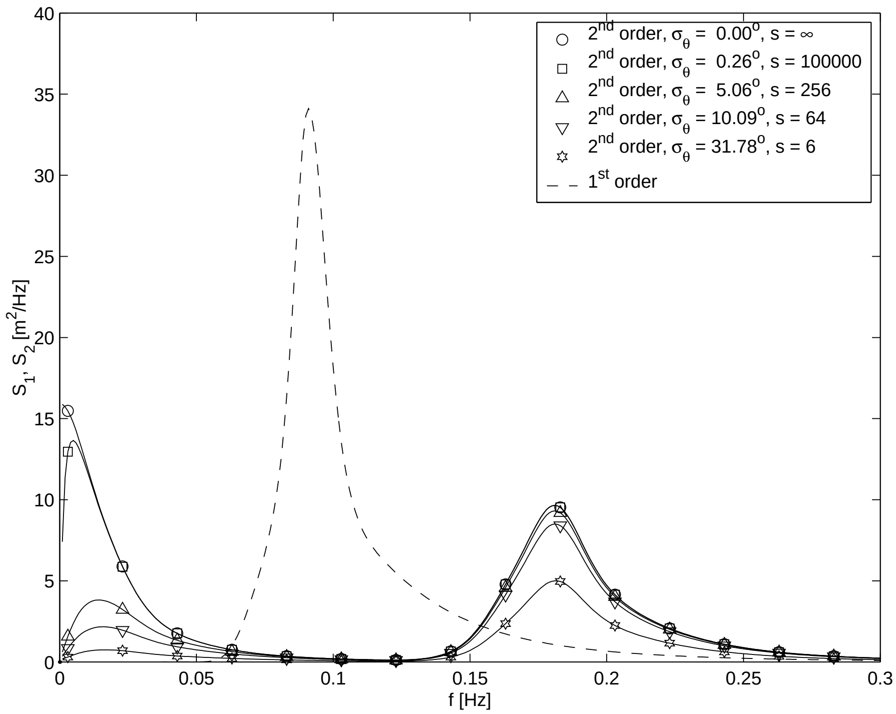

FIG. 4.1. *Influence of the amount of directional spreading on the second-order spectrum.*

The perturbation-series approaches make distinction at a certain frequency between free wave-components satisfying the wave dispersion relationship and bound components forming a non-resonant triad with two free components. The bound wave-components are traveling with a celerity not satisfying the wave dispersion relationship. This multi-component description per wave frequency and direction is different from the approaches described in Section 2 (*e.g.* Becq-Girard *et al.*, 1999; Eldeberky and Madsen, 1999; Herbers and Burton, 1997; Rasmussen, 1998) which consider only one wave component per wave frequency and direction.

Using the nomenclature of Laing (1986), the free surface elevation $\zeta(\boldsymbol{x},t)$ in the perturbation-series approach can be described as:

$$\begin{aligned}\text{(4.1a)}\qquad \zeta(\boldsymbol{x},t) = &\sum_{m=-\infty}^{\infty} A_m \exp[\, i\, (\omega_m t - \boldsymbol{k}_m \cdot \boldsymbol{x})] \; + \\ &+ \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{D_2(\omega_m, \boldsymbol{k}_m; \omega_n, \boldsymbol{k}_n)}{g\, [1 - \Omega_{m,n}^2/(g\, \kappa_{m,n}\, \tanh \kappa_{m,n}\, h)]}\, A_m\, A_n\, \exp[\, i\, (\Omega_{m,n} t - \boldsymbol{\kappa}_{m,n} \cdot \boldsymbol{x})],\end{aligned}$$

with $\Omega_{m,n} = \omega_m + \omega_n$ and $\boldsymbol{\kappa}_{m,n} = \boldsymbol{k}_m + \boldsymbol{k}_n$ respectively the angular frequency and the wave number of the bound-wave components with kernel D_2, $\kappa_{m,n} = |\boldsymbol{\kappa}_{m,n}|$, (ω_m, ω_n) and $(\boldsymbol{k}_m, \boldsymbol{k}_n)$ the angular frequency and wave number of the free-wave components. Further we have for the negative-indexed quantities the following symmetry relations in order to get a real-valued free surface elevation $\zeta(\boldsymbol{x},t)$:

$$\text{(4.2)}\qquad \omega_{-m} = -\omega_m, \qquad \boldsymbol{k}_{-m} = -\boldsymbol{k}_m, \qquad \text{and} \qquad A_{-m} = A_m^*.$$

Assuming the first-order wave components A_m to be due to a stationary Gaussian process, it is easy to derive an expression for the (two-sided) power spectrum $\Phi(\omega,\boldsymbol{k})$ of the free surface elevation $\zeta(\boldsymbol{x},t)$:

$$\text{(4.3a)}\qquad \Phi(\omega,\boldsymbol{k}) = \Phi_1(\omega,\boldsymbol{k}) + \Phi_2(\omega,\boldsymbol{k}),$$

$$\text{(4.3b)}\qquad \Phi_2(\omega,\boldsymbol{k}) = \int\!\!\int d\boldsymbol{k}_1\, d\omega_1\; D_{20}(\omega_1,\boldsymbol{k}_1;\omega-\omega_1,\boldsymbol{k}-\boldsymbol{k}_1)\; \Phi_1(\omega_1,\boldsymbol{k}_1)\; \Phi_1(\omega-\omega_1,\boldsymbol{k}-\boldsymbol{k}_1),$$

$$\text{(4.3c)}\qquad D_{20}(\omega_1,\boldsymbol{k}_1;\omega_2,\boldsymbol{k}_2) = 2\, \left\{ \frac{D_2(\omega_1,\boldsymbol{k}_1;\omega_2,\boldsymbol{k}_2)}{g\, [1 - \hat{\Omega}_{1,2}^2/(g\, \hat{\kappa}_{1,2}\, \tanh \hat{\kappa}_{1,2}\, h)} \right\}^2,$$

with $\Phi_1(\omega,\boldsymbol{k})$ the two-sided first-order spectrum of the free wave components, $\hat{\Omega}_{1,2} = \omega_1 + \omega_2$, $\hat{\kappa}_{1,2} = |\hat{\boldsymbol{\kappa}}_{1,2}|$ and $\hat{\boldsymbol{\kappa}}_{1,2} = \boldsymbol{k}_1 + \boldsymbol{k}_2$.

For waves with a first-order JONSWAP spectrum and $\cos^{2s}(\theta/2)$ directional distribution in a water depth of 8 meter and a peak frequency of 0.91 Hz, Figure 1 gives an example of the influence of the amount of direction spreading on the wave-number integrated one-sided frequency spectra $S_1(f)$ and $S_2(f)$:

$$S_1(f) = 4\pi \int \Phi_1(2\pi f, \boldsymbol{k})\, d\boldsymbol{k} \qquad \text{and} \qquad S_2(f) = 4\pi \int \Phi_2(2\pi f, \boldsymbol{k})\, d\boldsymbol{k}. \tag{4.4}$$

For a given first-order free-wave spectrum $\Phi_1(\omega, \boldsymbol{k})$ the associated second-order bound-wave spectrum $\Phi_2(\omega, \boldsymbol{k})$ can be determined in a straightforward manner. However, in general the total spectrum $\Phi(\omega, \boldsymbol{k})$ is given and $\Phi_1(\omega, \boldsymbol{k})$ and $\Phi_2(\omega, \boldsymbol{k})$ are unknown. Laing (1986) used an iterative procedure to split a given spectrum into a (first-order) free-wave part and a (second-order) bound-wave part.

5. Concluding remarks. In the coastal zone wave non-linearity is dominated by non-resonant and near-resonant interactions between wave triads. This in contrast with the situation in deep water, where the energy content of the bound sub-harmonics is neglegible and the bound super-harmonics also become much less important (but do not vanish).

Two approaches are being studied: cumulant-closure approaches and perturbation-series approaches. At the moment there exist no well-proven cumulant-closure relationships, while the perturbation-series approaches are self-contained at a certain order and do not need additional closure. At this moment the inclusion of a perturbation-series approach (*e.g.* Laing, 1986; Willebrand, 1975) into a spectral model for coastal regions like SWAN (Booij *et al.*, 1999) seems the most straight-forward method for the inclusion of the major non-resonant bound-wave triad interactions.

For a horizontal bed both the cumulant-closure approaches and the perturbation-series approaches are both supposed to include a description of second-order bound waves. So, a carefull comparison of both approaches for this horizontal bed case can lead to more insight into their working and into their weak and strong points.

Another point which still needs further research is the description of near-resonant triad interactions other than the bound waves.

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