Simulation of Sloshing Waves in a 3D Tank Based on a Pseudo-Spectral Method*

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Abstract

Sloshing waves in a three dimensional tank are modelled using a pseudo spectral method based on fully non linear potential theory. The formulation is based on the expansion of the velocity potential in series of the natural modes of the tank geometry. Mode coefficients of the potential and nodal values of the free surface elevation are determined by accounting for the fully non linear kinematic and dynamic free surface conditions. The theoretical quasi-exponential convergence of the model with respect to the number of modes is verified in the case of free oscillations in a fixed 2D rectangular tank. Further results are given for 2D or 3D tanks submitted to forced motions. These results are found to be in very good agreement with available data.

Introduction

Spectral methods are characterized by the expansion of the solution in terms of global functions. When orthogonal functions are used, it can be shown that the approximation error decreases faster than algebraically. This behavior is referred to as exponential, or spectral convergence. The counterpart of this attractive feature is mainly found in the limitation to simple domains. However, their computational performances are such that spectral methods are prevailing for large scale computations in certain areas of fluid dynamics. This is for example the case in numerical weather forecast. For a review of the application of spectral methods in general computational fluid dynamics, see Hussaini & Zang (1987), or Fornberg (1995). In applications to free surface inviscid flows, it is possible to use orthogonal functions satisfying Laplace's equation, so that coefficients of the spectral expansions are determined through the free surface conditions only. Fenton & Rienecker (1982) solved nonlinear 2D wave propagation problems using spectral expansions both for the potential and the free surface elevation, under the assumption of space periodicity. In Dommermuth & Yue (1987) , three dimensional wave problems were simulated using a spectral method based on a perturbation expansion of free surface conditions. In Kim *et al* (1998) , fully non linear simulations in 2D rectangular tanks of infinite depth were reported. In Chern *et al* (1999), a spectral method based on Chebyshev polynomials was applied for solving fully nonlinear 2D free surface problems in a rectangular 2D tank, with the advantage of a fixed computational domain obtained by applying a time-depending σ-transform to the vertical co-ordinate. In the present paper, a spectral approach is applied to fully non linear sloshing waves in 2D or 3D rectangular tanks, using the natural modes of the fluid domain as a basis for the spectral expansion, and solving the boundary value problem in the physical space.

Mathematical Formulation and Numerical Solution

We consider a three dimensional tank, partially filled with an inviscid fluid. A cartesian fixed co-ordinate system $0xyz$ is defined. Assuming potential flow, a problem for a scalar velocity potential Φ is set up. The potential has to satisfy Laplace's equation in the fluid domain, as well as Neumann conditions on the tank walls and bottom:

$$\Delta\Phi(M,t) = 0 \quad M \in D \tag{1}$$

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$$\frac{\partial \Phi}{\partial n} = U.n \quad on\, solid\, boundaries \tag{2}$$

In the present formulation, we suppose a single-valued free surface F, represented by $z = \eta(x,y,t)$. The kinematic and dynamic conditions at the free surface are thus formulated as follows (with implicit non-dimensionalization with respect to the mean water depth h and the acceleration of gravity g):

$$\frac{\partial \eta(x,y,t)}{\partial t} = \frac{\partial \Phi}{\partial z} - \frac{\partial \Phi}{\partial x}\frac{\partial \eta}{\partial x} - \frac{\partial \Phi}{\partial x}\frac{\partial \eta}{\partial x} \; ; \; M(x,y,z,t) \in F \tag{3}$$

$$\frac{\partial \Phi(M,t)}{\partial t} = -z - \frac{1}{2}|\overrightarrow{\nabla}\Phi|^2 \; ; \; M(x,y,z,t) \in F \tag{4}$$

Then, we introduce a spectral expansion of Φ in series of natural modes of the tank:

$$\Phi(x,y,z,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}(t)\Psi_{mn}(x,y,z) \tag{5}$$

where Ψ_{mn} are eigen functions of the fluid domain, satisfying equations (1) and (2), and a_{mn} are time depending modal amplitudes. Here we consider a parallelepipedic tank, for which eigen functions are given by:

$$\Psi_{mn}(x,y,z) = \frac{cosh(k_{mn}(z+1))}{cosh(k_{mn})} cos(\overrightarrow{k_{mn}}\,\overrightarrow{x}) \tag{6}$$

where $\overrightarrow{x} = (x,y)$ and $\overrightarrow{k_{mn}} = (m\pi/L_x, n\pi/L_n)$ is the wave number associated to each mode.

After truncation of the spectral expansion, equation (5) is fed in the dynamic free surface condition:

$$\sum_{m=0}^{N_x}\sum_{n=0}^{N_y} \frac{da_{mn}(t)}{dt}\Psi_{mn}(x,y,\eta) = -\eta(x,y,t) - \frac{1}{2}|\overrightarrow{\nabla}\Phi(x,y,\eta,t)|^2 \tag{7}$$

In the solution of wave propagation problems using a spectral approach, the assumption of space periodicity may be introduced to further expand the free surface elevation in spectral Fourier series [3]. Here for fully non linear sloshing problems, the free surface elevation cannot a priori be expanded in Fourier series of horizontal co-ordinates. Thus $\eta(x,y,t)$ is represented by its nodal values at free surface collocation points, on which free surface conditions are imposed.

Starting from given initial conditions, the initial boundary value problem is thus solved for N_Φ modal amplitudes of the potential and N_η nodal values of the free surface elevation. A standard 4th order Runge-Kutta scheme is applied for advancing the solution in time, by integrating the free surface conditions as ODE for a_{ij} and η_k.

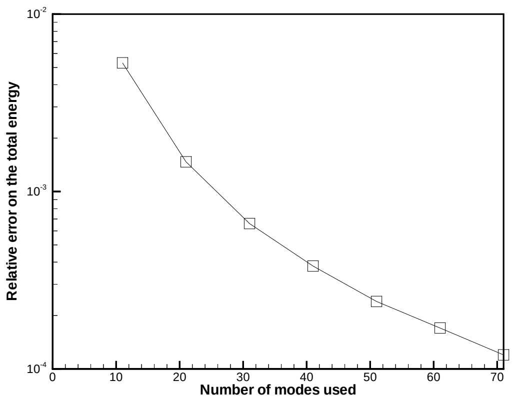
The system of first order differential equations for a_{ij} is obtained at each substep of the time marching procedure by solving a system of linear algebraic equations resulting from the application of the dynamic condition (7) at a sufficient number of collocation points on the free surface. This is in contrast with other schemes for inviscid free surface flows in which nodal potential values are updating by directly applying the dynamic condition. Derivatives of the potential appearing at the right-hand sides of the FSC's are computed from the spectral expansions, while finite difference formulas are applied for the derivatives of the wave elevation.

When the problem is solved on the basis of a perturbation expansion procedure, see e.g. [4], the resulting time-invariant kernel is the same at each order of the expansion, and a FFT procedure can be applied, with a $O(N_\Phi Log(N_\Phi))$ effort at each time step. In the present fully nonlinear scheme, the kernel of the linear system formed of values of $\Psi_{mn}(x,y,\eta)$ is solution-dependent and has to be reevaluated at each sub-step. A preconditioned GMRES iterative solver is applied for solution of linear systems. The resulting cost is $O(N_\Phi^2)$ for a well-conditioned system, i.e. when the number of iterations at convergence is only weakly dependent on the size of the problem. The other significant part of the computational effort is devoted to the assembly of the kernel and to the computation of potential derivatives, thus the global effort is also $O(N_\Phi^2)$.

Numerical Results

2D free motion

In this section we consider a 2D tank, with initial conditions $\eta(x,y,0) = \eta_0(x,y)$ and $\Phi(x,y,z,0) = 0$. The fluid is initially at rest, and the initial free surface profile corresponds to the first linearized eigen mode, with an initial steepness $(\eta_{max} - \eta_{min})/Lx = 10\%$. The tank length is $L_x/h = 2$. This simple test case aimed at a verification of the convergence properties of the scheme. Free motion simulations over 7 dominant periods were repeated with increasing number of modes and free surface nodes: $N_\Phi = N_\eta = 11, 21, 31, 41, 51, 61, 71$. The maximum difference between the instantaneous fluid energy and the initial (potential) energy has been computed in each case. Results are plotted in figure 1. The expected spectral convergence is obtained, with a max relative error of 10^{-4} on the energy with 71 modes. The error on the fluid volume is about 10^{-7}, and does not vary significantly with N_Φ. Free surface profiles corresponding at each successive maxima of the free surface elevation are collected in figure 2.



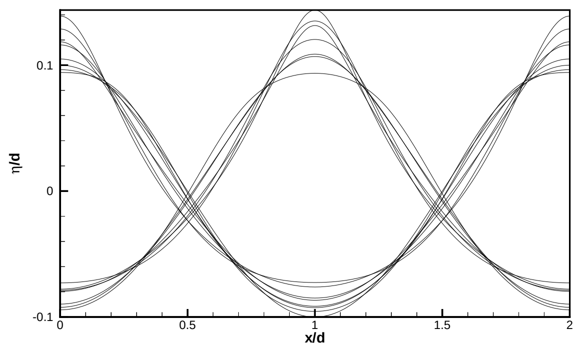


Figure 1 *(left): Convergence of the relative energy error with respect to number of modes*
Figure 2 *(right): Successive extrema of the free surface profile*

2D forced motions

Here we consider a 2D tank with fluid initially at rest, submitted to a forced motion $x(t) = a\, sin(\omega t)$. The problem formulation is modified to account for the moving co-ordinate system, following closely the approach of Wu *et al* (1998), in which nonlinear sloshing problems are solved using a finite element method. The tank length is $L_x/h = 25.$, a very shallow water case, for which strong nonlinear effects are anticipated. The reduced motion amplitude is $a/h = 2.5$. The angular frequency is $\omega = 0.9973\omega_0$, where ω_O is the frequency of the first linearized natural mode. This is just a case presented in [7]. The simulation has been run with 40 modes . Figure 3 below is the equivalent of figure 9 in [7], plotted with the results of our spectral model. A bore is formed at the end of the simulation (figure 3-b). Results from both approaches seem very similar up to the formation of the bore, at about $t = 26.$ (figure 3-a). Then results differ slightly. With the present spectral method, a steeper wave front followed by very short waves are exhibited. However, no convergence tests have been performed to date on this case. Such tests will be available for presentation at the workshop.

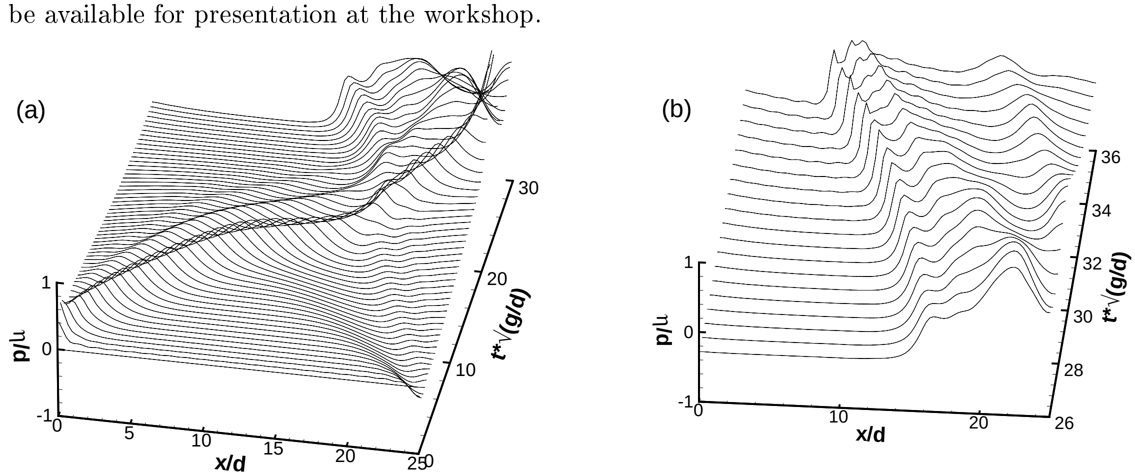
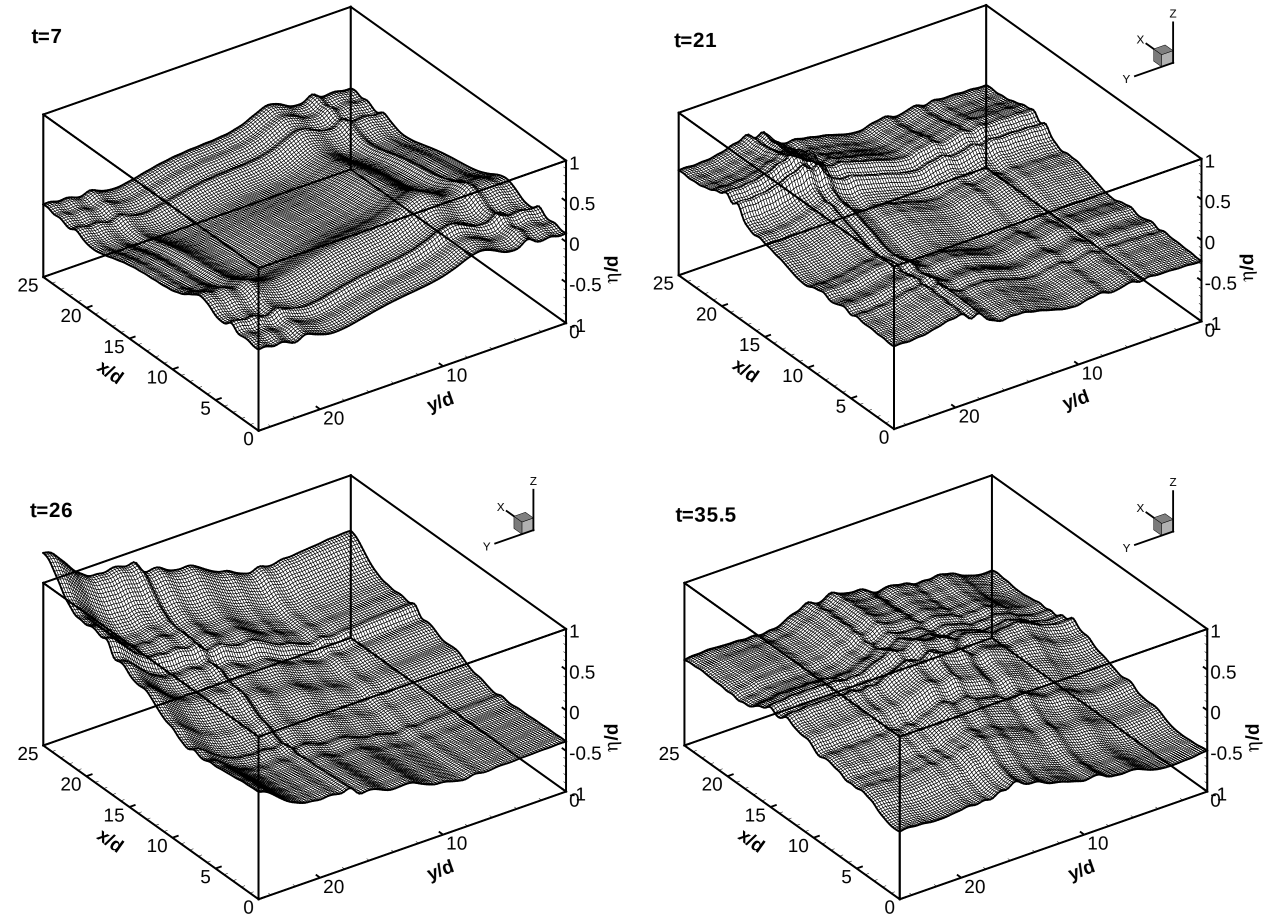


Figure 3*: Wave profiles for forced motions, $L_x/h = 25.$, $\omega = 0.9973\,\omega_0$, $a/h = 2.5$. (a): $t = 0 \sim 30$. (b): $t = 26 \sim 36$.*

3D forced motions

A shallow water case is now considered in the case of a 3D tank: $L_x/h = 25.$, $L_y/h = 25$. The parameters corresponding to case (H) in table 1 of Wu et al (1998) have been selected. The tank is subject to a sine motion along the first diagonal of the undisturbed free surface, $x = a_x\, sin(\omega_x t)$; $y = a_y\, sin(\omega_y t)$, with $a_x = a_y = 1.2h$ and $\omega_x = \omega_y = 0.998\omega_{0x}$. The simulation was performed with 1600 (40x40) modes. We give below four examples of free surface profiles in the tank. Longer simulations and comparisons with results given in [7] are underway and will be presented at the workshop, together with energy and volume checks.



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