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ON A FAST METHOD FOR SIMULATIONS OF STEEP WATER WAVES

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The dynamics of waves along the ocean surface plays an important role for safety of marine activity and operations. The waves determine the input parameters for dimensioning of oil-platforms and ships. Further, the waves determine the loads in tension-legs and risers connected to oil-platforms and floating production ships. The most common industrial analysis tools for waves at the sea surface (irregular waves) employ perturbation models, capturing nonlinear effects up to the second or the third order in wave steepness. Observations both in large scale and in laboratories reveal that weakly nonlinear methods have shortcomings in modelling moderately steep waves and the corresponding induced velocities and accelerations. Prominent examples are waves leading to ringing of offshore structures and highly nonlinear freak waves. Fully nonlinear methods which capture the weaknesses of weakly nonlinear methods have primarily been employed to study breaking waves. Here we focus an intermediate amplitude range, where perturbation models have poor performance, but the amplitudes are below those leading to breaking.

A common drawback of the existing fully nonlinear methods is that the computational schemes are slow. This means that long time simulations of wave-fields with appreciable size are unrealistic. While the integration of the prognostic equations can be made fast, the bottleneck is the solution of the Laplace equation which is required at each time step. Thus, a fully nonlinear model for water waves can only be fast provided that the Laplace equation solver is fast. Here the aim is to derive a rapid method for fully nonlinear non-overturning water waves. The formulation is two-dimensional, but the method may be extended also to the three-dimensional case. Making use of potential theory we introduce velocity potential and stream function (ϕ, ψ), and $(x,\ y,\ t)$ as horizontal, upward vertical and time variables, and let $\eta(x,t)$ be the surface elevation relative to the mean level $y=0$. In two dimensions we obtain ϕ and ψ by the Cauchy integral formula, split into real and imaginary parts, giving

$$\widetilde{\phi} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{D\,(\,\widetilde{\phi}' - \eta'_x\,\widetilde{\psi}'\,) - \widetilde{\psi}' - \eta'_x\,\widetilde{\phi}'}{1 + D^2}\,\frac{\mathrm{d}x'}{x' - x}, \tag{1}$$

$$\widetilde{\psi} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\widetilde{\phi}' - \eta'_x\,\widetilde{\psi}' + D\,(\widetilde{\psi}' + \eta'_x\,\widetilde{\phi}')}{1 + D^2}\,\frac{\mathrm{d}x'}{x' - x}, \tag{2}$$

where the 'tildes' denote the functions at $y=\eta$ and $\widetilde{\phi}=\widetilde{\phi}(x,t)$, $\widetilde{\phi}'=\widetilde{\phi}(x',t)$, etc. In (1)–(2) the function $D=(\eta'-\eta)/(x'-x)$ is introduced, where D decays according to $|x'-x|^{-1}$ for $|x'-x|\to\infty$ and $D\to\eta_x$ for $x'\to x$. The equation (2) is commonly used to determine $\widetilde{\psi}$, given $\widetilde{\phi}$ and η. $\widetilde{\psi}$ is then determined implicitly, and the equation is typically solved iteratively with

$\mathcal{O}(N^2)$ operations. This is the intensive part of the computations. An alternative, however, is to determine $\widetilde{\psi}$ from equation (1).

When the surface is horizontal, the integral equations are convolution products and can therefore be computed very quickly *via* Fast Fourier Transform. For a non-horizontal surface it is then tempting to reformulate the equation obtaining the form of convolutions. Splitting (1) into singular and regular integrals we obtain after one integration by parts

$$\widetilde{\phi} = -\frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{\widetilde{\psi}'}{x'-x}\,\mathrm{d}x' + \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{\eta'\,\widetilde{\phi}'_x}{x'-x}\,\mathrm{d}x' - \frac{\eta}{\pi} \fint_{-\infty}^{\infty} \frac{\widetilde{\phi}'_x}{x'-x}\,\mathrm{d}x' \\ + \frac{1}{\pi} \int_{-\infty}^{\infty} [\,\arctan(D) - D\,]\,\widetilde{\phi}'_x\,\mathrm{d}x' + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{D\,(D-\eta'_x)\,\widetilde{\psi}'}{1+D^2}\,\frac{\mathrm{d}x'}{x'-x}. \tag{3}$$

Applying the Hilbert transform (i.e. $\mathcal{H}\{f\} \equiv \frac{1}{\pi}\fint_{-\infty}^{\infty} \frac{f(x')}{x'-x}\,\mathrm{d}x'$), equation (3) becomes

$$\widetilde{\psi} = \mathcal{H}\left\{\widetilde{\phi}\right\} + \eta\,\widetilde{\phi}_x + \mathcal{H}\left\{\eta\,\mathcal{H}\left\{\widetilde{\phi}_x\right\}\right\} \\ - \mathcal{H}\left\{\frac{1}{\pi} \int_{-\infty}^{\infty} [\,\arctan(D) - D\,]\,\widetilde{\phi}'_x\,\mathrm{d}x' + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{D\,(D-\eta'_x)\,\widetilde{\psi}'}{1+D^2}\,\frac{\mathrm{d}x'}{x'-x}\right\}. \tag{4}$$

This is another equation for $\widetilde{\psi}$. In (4), the singular integrals are convolutions and can thus be computed quickly. The remaining regular integrals have kernels that decrease rapidly, as $|x'-x|^{-3}$ and $|x'-x|^{-2}$, respectively. Therefore, integrations over $(-\infty, +\infty)$ can be approximated by integrations over a limited interval $(x-\lambda, x+\lambda)$. The parameter λ is choosen in accordance with the precision needed and depends on the wave characteristics and not on the length of the computational domain. Moreover, the contribution on the right hand side of (4) involving $\widetilde{\psi}$, is cubic in nonlinearity, while in equation (2) the corresponding term is quadratic. For nonbreaking waves, iterations with (4) thus converge faster than iterations with (2). The convergence is so fast that one iteration is enough for most of the practical computations (see below).

An iterative scheme is initialized by the explicit quadratic approximation

$$\widetilde{\psi}_1 = \mathcal{H}\left\{\widetilde{\phi}\right\} + \eta\,\widetilde{\phi}_x + \mathcal{H}\left\{\eta\,\mathcal{H}\left\{\widetilde{\phi}_x\right\}\right\}. \tag{5}$$

Applying one analytical iteration, neglecting integrals being of quartic nonlinearity, we get another approximation

$$\widetilde{\psi}_{2,\lambda} = \widetilde{\psi}_1 - \mathcal{H}\left\{\frac{1}{\pi} \int_{x-\lambda}^{x+\lambda} \frac{D\,(D-\eta'_x)\,\widetilde{\psi}'_1}{1+D^2}\,\frac{\mathrm{d}x'}{x'-x}\right\}. \tag{6}$$

The latter is explicit and does not involve transcendental functions. It is very accurate and quickly computable. Integration over one wavelength of steady periodic waves, with almost maximal slopes, cannot be distinguished from reference computations (fig. 1).

The formulation is also tested in unsteady simulations with the following method. The linear parts of the temporal evolution equations are solved analytically, while the remainding nonlinear parts are solved numerically with a variable step-size eight-order explicit Runge–Kutta scheme. A spectral method is used to compute the spacial derivatives, without smoothing or regridding. Evolution of a long wave packet, with initially small slope of the carrier wave ($ak_0 = 0.12$), is simulated. Very large waves — freak waves — are formed after some while (fig. 2). The omitted term in (4) is quartic in nonlinearity with a small coefficient, and thus very small. This term may easily be included if needed. For most of the simulations it may be neglected. Therefore, the method is fully nonlinear in practice.

The method is $\mathcal{O}(N \log N)$ for practical computations, and thus very fast. Extensions include the three dimensional case and a finite (varying) fluid depth (Clamond & Grue 2001).

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References

Clamond, D. & Grue, J. 2001. A fast method for fully nonlinear water wave computations. Under consideration for publication in *J. Fluid Mech.*

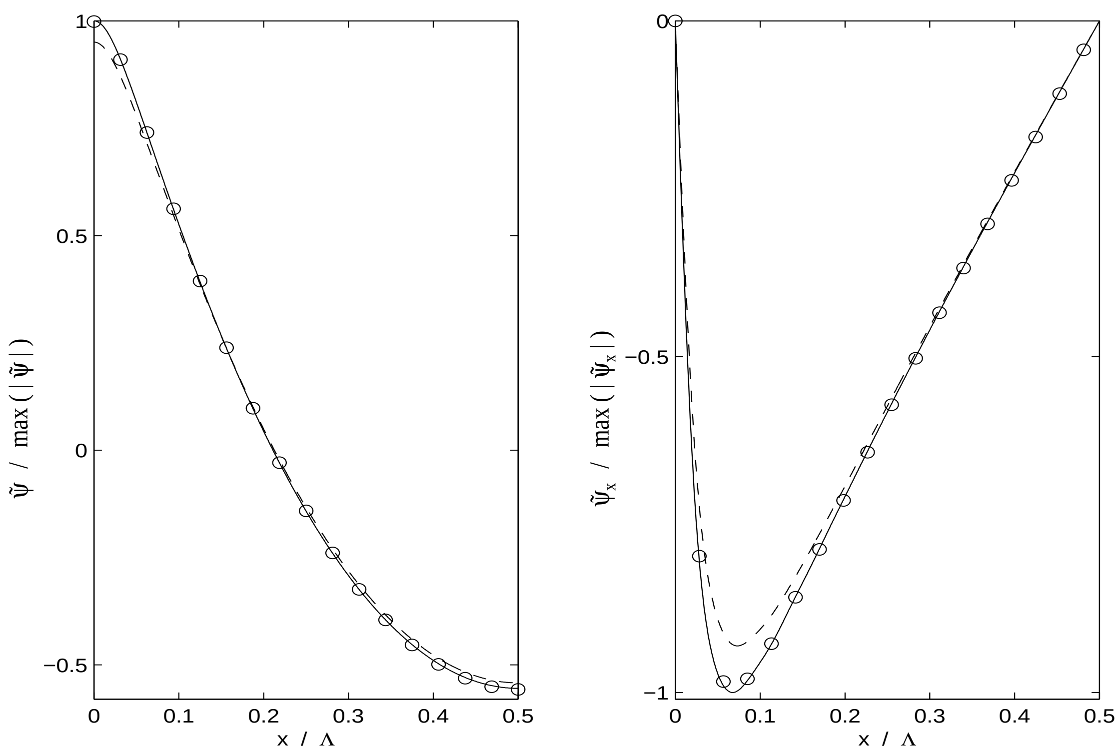

Figure 1: *Comparison of approximations for $2\pi a/\Lambda = 0.41$ (Λ wavelength).*

— exact, – – $\widetilde{\psi}_1$ (eq. 5), o $\widetilde{\psi}_{2,\lambda}$ (eq. 6 with $\lambda = \Lambda/2$).

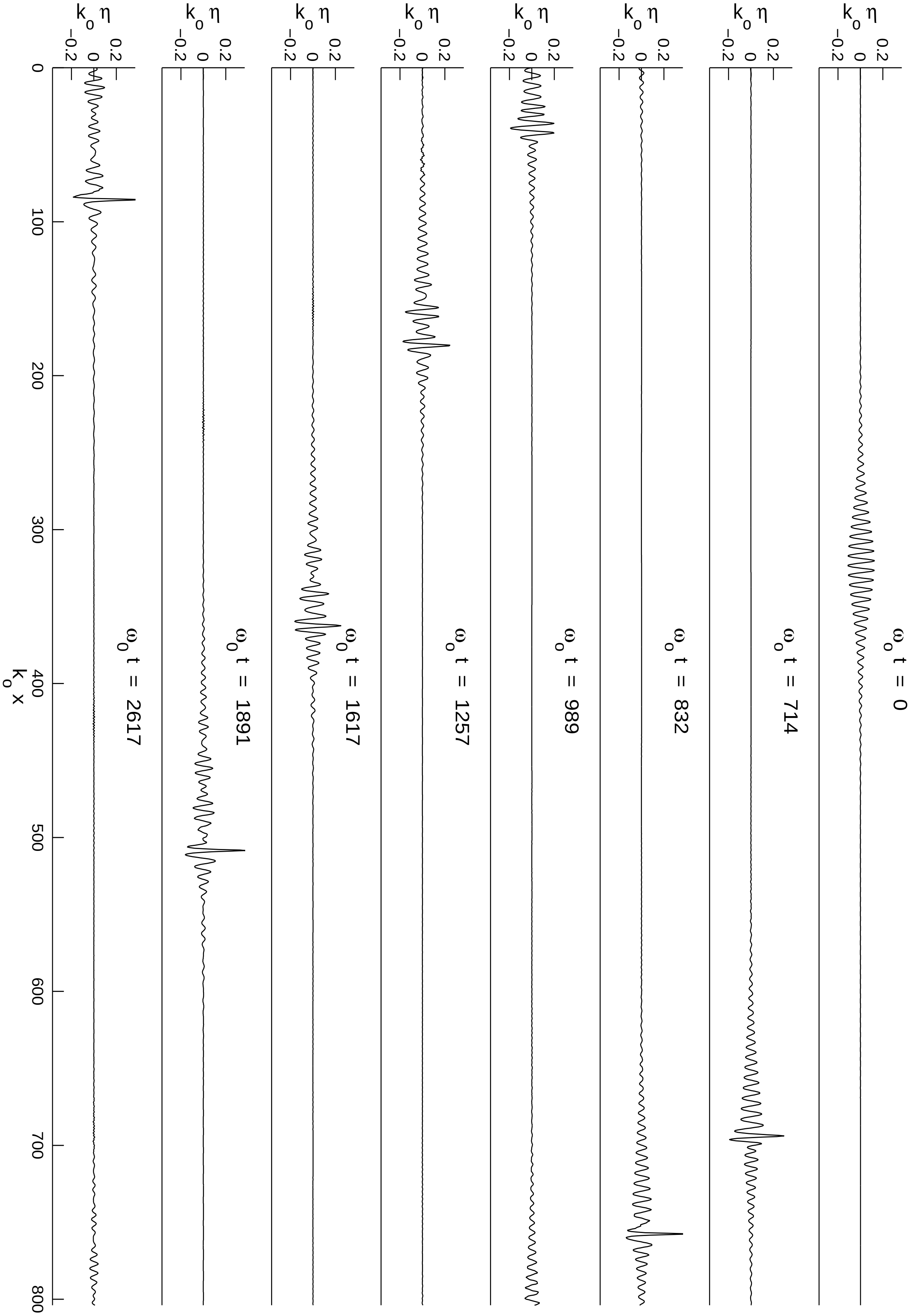

Figure 2: *Temporal evolution of a wave packet.*