Third-harmonic Diffraction Force on Axisymmetric Bodies

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1 Introduction

It was observed in model tests and prototype experiments that tension leg platforms(TLPs) and gravity base structure(GBSs) experience sudden bursts of highly amplified resonant activities (ringing) during storms. The ringing phenomenon will induce extreme stress in tethers, end even tethers breaking. It was found that ringing occurs at low frequency and ringing periods are about 3-5 times of the period of the corresponding incident waves. Thus, the calculation of third order force will be significant in predicting ringing phenomenon.

Nonlinear problems are characterized by forcing term in their boundary conditions. For the third order potential, the forcing term on the free surface includes both first and second order potentials. The difficulty in calculating third order force is that second order potential can not be expressed in an explicit form efficiently. Usually, it is represented by an integral equation and an infinite integration has to be carried out on the whole free surface. The present work proposed a one-step forward prediction method to calculate the second order potential on the free surface. Special concerns are also given to the treatment for the logarithmic singularity in the ring Green functions. Then, the third order forces are calculated by an indirect method, which is analogous to the indirect method for second order force.

The method has been implemented for axisymmetric bodies, and no difficulty has been found for extending it to arbitrary bodies, like TLPs. For axisymmetric bodies, a novel integral equation is also proposed.

2 Free Surface Condition

We assume the incident monochromatic waves have an incident frequency ω. Then , the first, second and third order harmonic potentials with the frequencies of $\omega_1 = \omega, \omega_2 = 2\omega$ and $\omega_3 = 3\omega$ are considered for the present interest. We separate the time dependencies explicitly, and write potentials at each order of ϵ as

$$\Phi^{(j)}(x,y,z,t) = \Re[\phi^{(j)}(x,y,z)e^{-i\omega_j t}] \tag{1}$$

Then, we can write the free surface conditions for each velocity potentials as

$$\begin{aligned} \nu_j\phi^{(j)} + \phi_z^{(j)} &= q^{(j)} \quad j=1,2,3,\cdots \quad \text{on } z=0 \\ \nu_j &= \omega_j^2/g \end{aligned} \tag{2}$$

where the forcing terms [2)] at each order of ϵ are

$$q^{(1)} = 0 \tag{3}$$

$$\begin{aligned} q^{(2)} &= -\frac{i\omega}{2g}\phi^{(1)}(-\frac{\omega^2}{g}\phi_z^{(1)} + \phi_{zz}^{(1)}) + \frac{i\omega}{g}\nabla\phi^{(1)}\cdot\phi^{(1)} \\ q^{(3)} &= \frac{3i\omega}{g}\nabla\phi^{(1)}\cdot\phi^{(2)} - \frac{i\omega}{2g}\phi^{(1)}(\phi_{zz}^{(2)} - 4\nu\phi_z^{(2)}) - \frac{i\omega}{g}\phi^{(2)}(\phi_{zz}^{(1)} - \nu\phi_z^{(1)}) \qquad (4) \\ &- \frac{1}{8g}\nabla\phi^{(1)}\cdot\nabla(\nabla\phi^{(1)}\cdot\nabla\phi^{(1)}) - \frac{\nu}{g}\phi^{(1)}\nabla\phi^{(1)}\cdot\nabla\phi_z^{(1)} \\ &+ \frac{1}{g}(\frac{\nu}{4}\phi^{(1)}\phi_z^{(1)} + \frac{1}{8}\nabla\phi^{(1)}\cdot\nabla\phi^{(1)})(\phi_{zz}^{(1)} - \nu\phi_z^{(1)}) \qquad (5) \end{aligned}$$

3 Integral Equation

We separate velocity potentials and forcing terms into incident and diffraction components as

$$\phi^{(j)} = \phi_I^{(j)} + \phi_D^{(j)}, \quad q^{(j)} = q_I^{(j)} + q_D^{(j)}$$

Expanding the diffraction potential and the Green function into the following series

$$\phi_D^{(j)}(\mathbf{x}) = \sum_{m=0}^{\infty} \epsilon_m \phi_{Dm}^{(j)}(r) \cos m\theta \tag{6}$$

$$G(\mathbf{x}, \mathbf{x}_0) = \sum_{m=0}^{\infty} \epsilon_m G_m(r, z; r_0, z_0) \cos m(\theta - \theta_0) \tag{7}$$

for axisymmetric body, we can derive the integral equation for the mth mode of jth order potential as

$$\begin{aligned}[\frac{1}{2\pi} - \nu_j \int_{\Gamma_W} G_0 r dr]\phi_{Dm}^{(j)}(r_0) - \int_{\Gamma_B} [\frac{\partial G_0}{\partial n}\phi_{Dm}^{(j)}(r_0) - \frac{\partial G_m}{\partial n}\phi_{Dm}^{(j)}(r)] r dl \\ = \int_{\Gamma_B} G_m \frac{\partial \phi_{Im}^{(j)}}{\partial n} r dl + \int_a^{\infty} G_m q_{Dm}^{(j)}(r) r dr\end{aligned} \tag{8}$$

after using a technique to weaken the singularity, where G_m is well known as the ring Green function and G_0 is the simple Green function which satisfies only the fixed free and bottom surface conditions.

4 Numerical Implement

For second order potential in fluid domain, two integrations have to be carried out both on body surface and on free surface when applying integral equation method.

4.1 Integral on the free surface

$$I_{Fm}(r_0, \theta_0, 0) = -\int_a^{\infty} r dr q_{Dm}^{(2)}(r)[i\pi C_0 H_m(k_2 r_>) J_m(k_2 r_<) + 2\sum_{n=1}^{\infty} C_n K_m(\kappa_n r_>) I_m(\kappa_n r_<)] \tag{9}$$

Defining L_{mn} term as

$$L_{mn} = 2C_n K_m(\kappa_n r_>) I_m(\kappa_n r_<) Z_n(\kappa_n z) \tag{10}$$

The limitation of L_{mn} term for large n is L_{mn}^s, and its infinite sum is L_m^s, which has a logarithmic singularity when the field point is close to the source point.

To remove the logarithmic singularity, we rewrite the integral as

$$\begin{aligned}&\sum_{n=1}^{\infty}\int_a^{\infty} L_{mn}(r) q_{Dm}^{(2)}(r) r dr \\ \approx &\sum_{n=1}^{N}\int_a^{\infty} L_{mn}(r) q_{Dm}^{(2)}(r) r dr - \sum_{n=1}^{N}\int_{r_0-\Delta r}^{r_0+\Delta r} L_{mn}^s(r) q_{Dm}^{(2)}(r) r dr + \int_{r_0-\Delta r}^{r_0+\Delta r} L_m^s(r) q_{Dm}^{(2)}(r) r dr\end{aligned} \tag{11}$$

For large modes, inside the range $(r - \Delta r, r + \Delta r)$ the first two terms can be canceled each other; outside the range the difference can be neglected. For the third term, a transform is used to remove the singularity. Then the integration can be represented as

$$I_{Fm}(r_0, 0) = -[\sum_{n=0}^{N} C_n T_{mn}(r_0) - \sum_{n=1}^{N} V_{mn}(r_0) + V_{m0}(r_0)] \tag{12}$$

where

$$V_{m0}(r_0) = -\frac{1}{2\pi}\int_{r_0-\Delta r}^{r_0+\Delta r} q_{Dm}^{(2)}(r) ln[1 - \exp(-\frac{\pi}{d}|r - r_0|)]\sqrt{\frac{r}{r_0}} dr \tag{13}$$

$$V_{mn}(r_0) = \frac{1}{2n\pi}\int_{r_0-\Delta r}^{r_0+\Delta r} q_{Dm}^{(2)}(r) \exp(-\frac{\pi}{nd}|r - r_0|)\sqrt{\frac{r}{r_0}} dr \tag{14}$$

$$T_{m0}(r_0) = S_{m01}(r_0) J_m(k_2 r_0) + S_{m02}(r_0) H_m(k_2 r_0) \tag{15}$$

$$T_{mn}(r_0) = S_{mn1}(r_0) e^{-\kappa_n r_0} I_m(\kappa_n r_0) + S_{mn2}(r_0) e^{\kappa_n r_0} K_m(\kappa_n r_0) \tag{16}$$

where

$$S_{m01}(r_1) = \frac{i\pi}{2}\int_{r_1}^{\infty} q_{Dm}^{(2)}(r) H_m(k_2 r) r dr = S_{m01}(r_0) - \frac{i\pi}{2}\int_{r_0}^{r_1} q_{Dm}^{(2)}(r) H_m(k_2 r) r dr$$

$$S_{m02}(r_1) = \frac{i\pi}{2}\int_{a}^{r_1} q_{Dm}^{(2)}(r) J_m(k_2 r) r dr = S_{m02}(r_0) + \frac{i\pi}{2}\int_{r_0}^{r_1} q_{Dm}^{(2)}(r) J_m(k_2 r) r dr$$

$$S_{mn1}(r_1) = \int_{r_1}^{\infty} q_{Dm}^{(2)}(r)K_m(\kappa_n r)rdr = S_{mn1}(r_0)e^{\kappa_n(r_1-r_0)} - \int_{r_0}^{r_1} q_{Dm}^{(2)}(r)K_m(\kappa_n r)rdre^{\kappa_n r_1}$$

$$S_{mn2}(r_1) = \int_{a}^{r_1} q_{Dm}^{(2)}(r)I_m(\kappa_n r)rdr = S_{mn2}(r_0)e^{-\kappa_n(r_1-r_0)} + \int_{r_0}^{r_1} q_{Dm}^{(2)}(r)I_m(\kappa_n r)rdre^{-\kappa_n r_1}$$

4.2 Integration on the body surface

For those points not close to body surface, we write the body integration as

$$I_{Bm}(r_0,0) = \int_{\Gamma_B} [\phi_{Dm}^{(2)}(r,z)\frac{\partial G_m}{\partial n} + \frac{\partial \phi_{Im}^{(2)}(r,z)}{\partial n}G_m]rdl = U_{m0}H_m(k_2 r_0) + \sum_{n=1}^{\infty} U_{mn}K_m(\kappa_n r_0) \quad (17)$$

where U_{m0} and U_{mn} are determined by the body integration. When the point is close to the body surface, a technique is also used to weaken the near singularity.

5 Hydrodynamic Force

We divide the third order force into three terms

$$F^{(3)} = \Re[(f_1^{(3)} + f_2^{(3)} + f_3^{(3)})e^{-3i\omega t}] \quad (18)$$

where $f_1^{(3)}, f_2^{(3)}$ and $f_3^{(3)}$ are

$$f_1^{(3)} = -\frac{i\omega}{8g}\rho \int_{C_B} [\phi^{(1)}\nabla\phi^{(1)}\cdot\nabla\phi^{(1)} + \nu^2(\phi^{(1)})^3]\mathbf{n}dc \quad (19)$$

$$f_2^{(3)} = -\frac{1}{2}\int\int_{S_B} \rho\nabla\phi^{(1)}\cdot\nabla\phi^{(2)}\mathbf{n}ds - \rho\nu\int_{C_B}\phi^{(1)}\phi^{(2)}\mathbf{n}dc \quad (20)$$

$$f_3^{(3)} = \int\int_{S_B} 3i\omega\rho[\phi_D^{(3)} + \phi_I^{(3)}]\mathbf{n}ds \quad (21)$$

Similar to the second order force, the third order diffraction force can be further divided as

$$f_{3j}^{(3)} = 3i\omega\rho\int\int_{S_B}[\phi_I^{(3)}n_j + \psi_j\frac{\partial\phi_I^{(3)}}{\partial n}]ds + 3i\omega\rho\int\int_{S_F}\psi_j q_D^{(3)}ds \quad (22)$$

by using an auxiliary radiation potential ψ_j at triple frequency of incident wave.

Dividing the third order forcing term $q^{(3)}$ into two terms, $q_A^{(3)}$(consisting of only the first order potential) and $q_B^{(3)}$ (consisting of the first and second order potentials), the most difficult integral on the free surface, i.e. $\int\int \psi_j q_{BD}^{(3)}ds$, can be represented as

$$\int\int_{S_F}\psi_j q_{BD}^{(3)}ds = \int\int_{S_F}[\psi_j(W - W_I - \frac{i\omega}{2g}\phi_D^{(1)}\phi_{Izz}^{(2)} - \frac{i\omega}{2g}(2\nabla_0\phi^{(1)}\cdot\nabla_0\phi_D^{(2)} - \phi_D^{(2)}\phi_{zz}^{(1)})) - \frac{i\omega}{2g}\psi_{jzz}\phi^{(1)}\phi_D^{(2)}]ds + \frac{i\omega}{2g}\oint_{C_B}[\phi^{(1)}\phi_D^{(2)}n_j + \psi_j\phi^{(1)}\frac{\partial\phi_I^{(2)}}{\partial n}]dl \quad (23)$$

using some transforms, where $\nabla_0 = (\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta})$ and

$$W = \frac{i\omega}{g}[3\nabla_0\phi^{(1)}\cdot\nabla_0\phi^{(2)} + 5\nu\phi^{(1)}\phi_z^{(2)} - \phi^{(2)}(\phi_{zz}^{(1)} - \nu^2\phi^{(1)})]$$

6 Numerical results

As the first step, a case of uniform cylinder with radius of a in a water depth of $d/a = 10$ is considered. Figure 2 shows the comparison of third order surge force of the present calculation with the Malenica and Molin's (M & M's), Faltinsen, Newman and Vinje's (FNV's) analytical solutions and experimental results. It can be seen that the present calculation agrees well with M & M's results except for f_{33} component.

Figure 3 shows the comparison of third order pitch moment about the free surface with experimental results. The experimental results are very scattered, and no conclusion can be found. From the calculation results it can be seen that there is a peak at low frequency. This maybe is the exciting source of ringing phenomenon on TLP and GBS. At that frequency, the corresponding force is not very high. The reason for this peak is the acting point of third order surge force is low.

References

1) Faltinsen, O.M., Newman, J.N. & Vinje,T.: Nonlinear loads on a slender vertical cylinder, *J.Fluid Mech.*, Vol.289, pp.179-198, 1995.

2) Malenica, S and Molin,B.: Third -harmonic wave diffraction by a vertical cylinder, *J. Fluid Mech.*, Vol.302, pp.203-229, 1995.

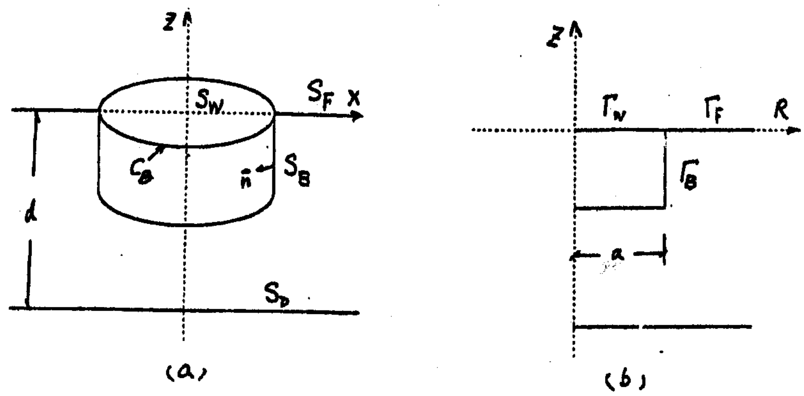

Fig 1 Definition of Sketch

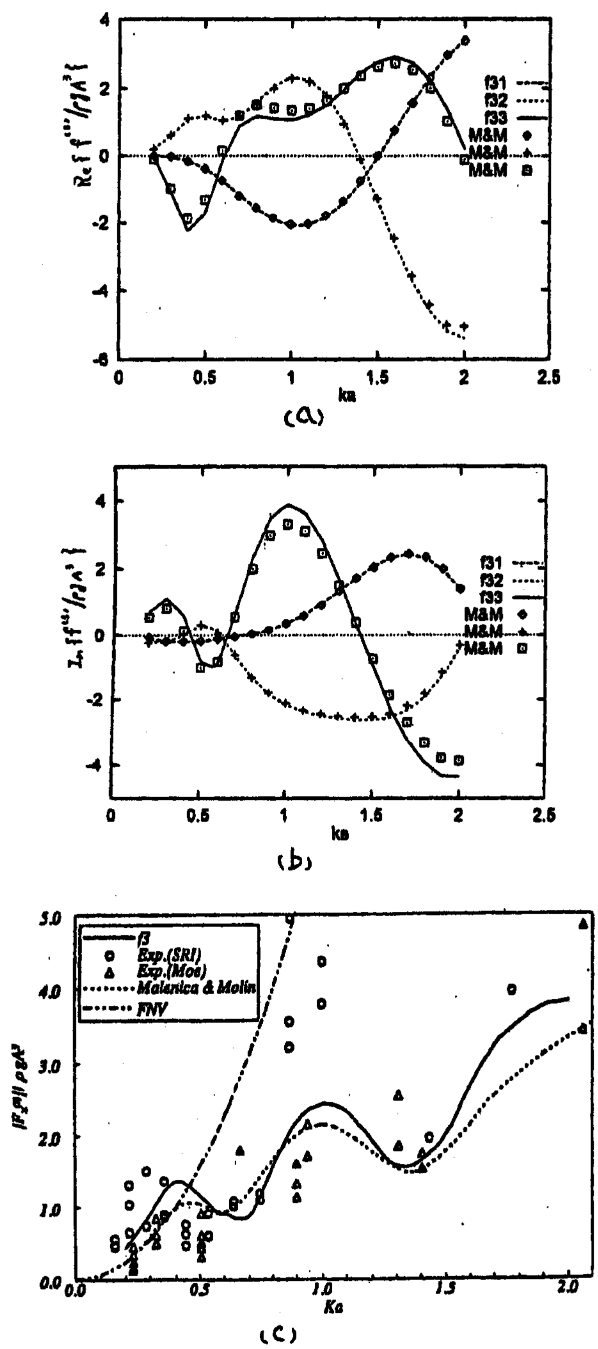

Figure 2 Third order surging force on a uniform cylinder (*d/a=10*)

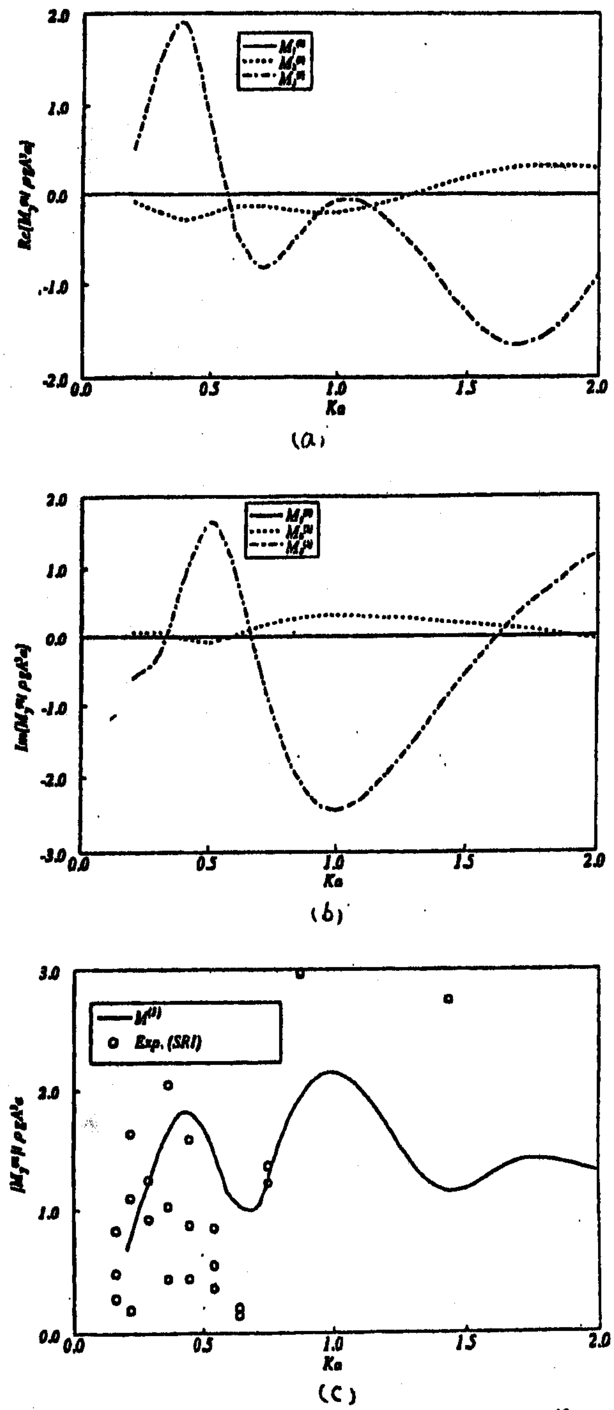

Figure 3 Third order pitching moment about Zc=0 on a uniform cylinder (d/a=1

DISCUSSION

Kim M.H.: The convergence test of the second-order diffraction computation is very complicated and cumbersome because of multiple parameters to be tested. Have you done any systematic convergence tests for the 3rd order diffraction problem?

Teng B., Kato S.: We did make systematic convergence tests for the 3rd order diffraction problem. At first I found our results were not correct. Then I spent a lot of effort on the examination of its convergence. Firstly, I examined the convergence of each terms. After having gotten their convergence, I made convergence test for the whole system.

Rainey R.C.T.: The authors are to be commended on the skill of their investigation of the difficult topic of 3rd order diffraction. However, I dispute its relevance to "ringing" because:

1) in waves big enough to cause "ringing", the ratio (wave height)/(cylinder diameter) is generally greater than 1, so that Stokes's expansion has diverged.

2) "ringing" can be at much higher frequencies than the 3rd harmonic, e.g. at 10 times the wave frequency.

See the forthcoming paper by J. Chaplin et al. (*Journal of Fluid Mechanics, 1997*).

Teng B., Kato S.: This study deals with the estimation method of 3rd harmonic forces on axisymmetric floating bodies by means of straight forward perturbation technique.

We do not claim that the cause of ringing of a uniform cylinder is due to 3rd harmonic wave forces.