Resonance in the unbounded water wave problem

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Introduction

In recent work, McIver (1996) demonstrated that there exist configurations of bodies in two dimensions for which the linearised water wave problem does not have a unique solution at a certain frequency. Non-uniqueness was proved by establishing the existence of a non-zero solution to the homogeneous boundary value problem at the relevant frequency. Such a solution, which decays at infinity and has finite energy, is called a *trapped mode.* In this work, it is shown that for such bodies the solution to certain forced boundary value problems, such as the heave problem, do not exist at the trapped mode frequency. The non-existence of the heave potential means that there is no steady state solution to the problem in which the bodies are forced to make small vertical oscillations about their mean position and similar interpretations may be made of the non-existence of other potentials. In related problems in waveguides, Werner (1987) has shown that trapped modes are closely related to resonances in certain initial value problems. By investigating the solution to a specific initial value problem in which an oscillatory pressure forcing is given to the free surface, it is shown that the steady state potential does not exist because resonance occurs. Work is currently underway to investigate the links between trapped modes and resonances in the more general initial value problem, using the spectral theory techniques of Goldstein (1969).

Construction of the trapped mode potential

An example of a system of bodies for which non-uniqueness occurs is found by constructing a potential which decays to zero at infinity and interpreting some of its streamlines as body boundaries. The potential is constructed by placing two infinite depth wave sources in the free surface, separated by a distance of half a wavelength. (Other suitable potentials may be constructed from sources placed any odd number of half wavelengths apart or a source and a sink placed an even number of wavelengths apart.) Thus the potential is given by

$$\phi = \int_0^\infty \frac{e^{-ky}}{k-K} \cos k(x-a)\, dk + \int_0^\infty \frac{e^{-ky}}{k-K} \cos k(x+a)\, dk, \tag{1}$$

where the contour of integration passes below the pole in each integral and $Ka = \pi/2$. Coordinate axes are chosen so that the origin is in the mean free surface and the $y-$axis points vertically downwards. The distance between the sources is $2a$ and $K = \omega^2/g$, where ω is the angular frequency and g is the acceleration due to gravity. As the sources are half a wavelength apart, the waves produced by each source cancel at either infinity and the potential decays to zero. In addition, the contributions to the integrals arising from the integration below the poles cancel and so the potential is purely real. The streamlines for the potential are illustrated in figure 1, where the variables have been nondimensionalised so that $x' = Kx$ and $y' = Ky$ and so the sources are at the positions $(\pm\pi/2, 0)$.

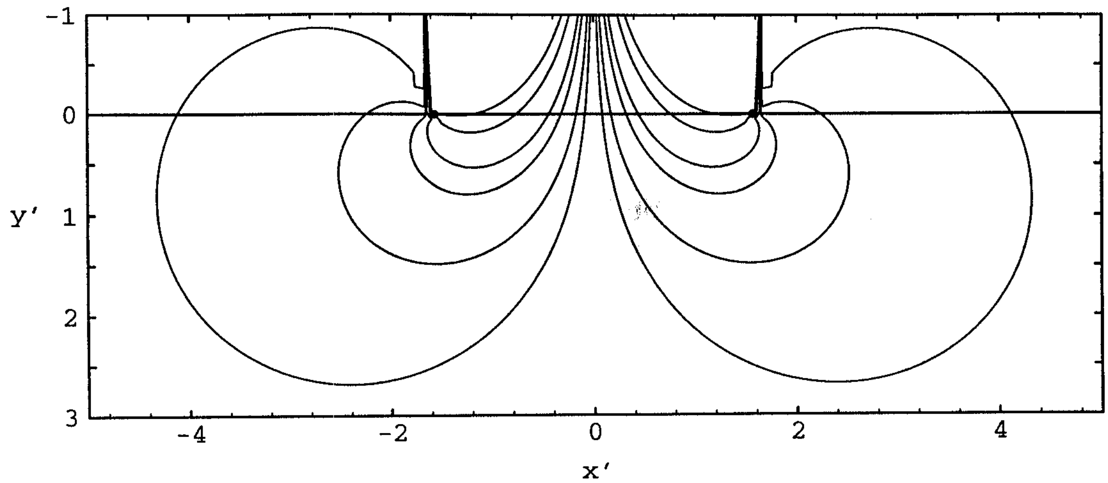

Figure 1 - The streamline pattern for two wave sources a nondimensional distance π apart

The half-plane $y' < 0$ represents the region above the free surface and so the potential has no physical meaning there but the continuation of the streamlines into that region has been included to illustrate how the streamlines cross the free surface. Any parts of streamlines which remove the singularities from the fluid may be chosen to represent body boundaries and an example of such bodies is given in figure 2.

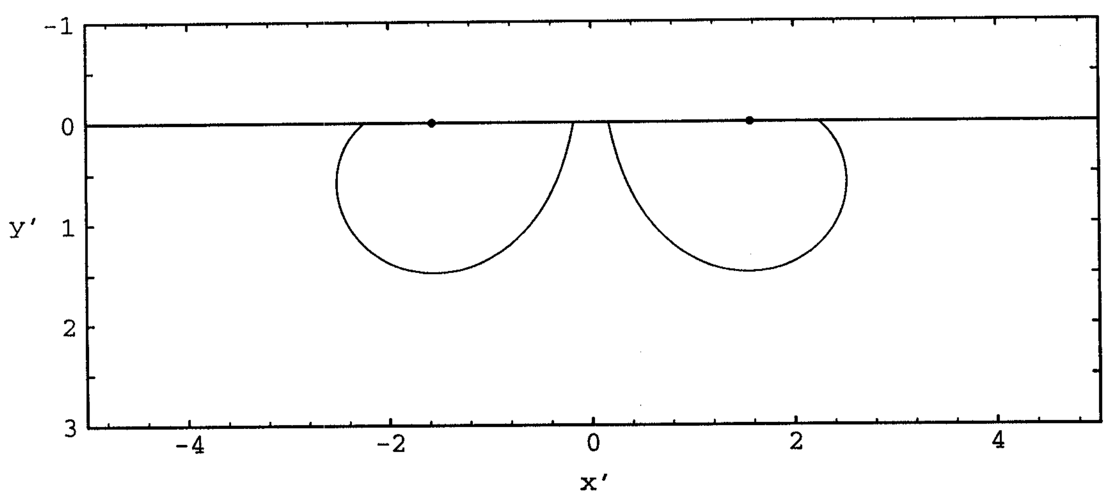

Figure 2 - Two surface-piercing bodies for which non-uniqueness occurs

For the remainder of this work attention will be confined to a pair of bodies such as those given in figure 2 for which a trapped mode exists at frequency $\omega = \omega_0$ (wavenumber $K = K_0$). The trapped mode potential will be denoted by $\phi_0(x, y)$.

Non-existence of the heave potential

The boundary value problem for the heave potential $\phi_h(x, y)$ at frequency ω_0 is the same as that for the trapped mode potential ϕ_0 except that ϕ_h produces outgoing waves at infinity and the normal derivative of ϕ_h on the bodies is given by

$$\frac{\partial \phi_h}{\partial n} = n_y, \tag{2}$$

where n_y denotes the component of the inward normal to the bodies in the vertical direction. Under the assumption that ϕ_h exists, an application of Green's theorem to ϕ_0 and ϕ_h in the fluid region outside the bodies gives

$$\int_{\text{bodies}} \phi_0 \, n_y \, dS = 0. \tag{3}$$

Thus equation (3) is a necessary condition for the existence of ϕ_h. (It is equivalent to the orthogonality condition which appears in the Fredholm alternative.) However, an application of Green's theorem to ϕ_0 and $u = y - 1/K_0$ in the fluid region outside the bodies, closed by a large semicircle, shows that

$$\int_{\text{bodies}} \phi_0 \, n_y \, dS = -\pi K_0 p_1. \tag{4}$$

where p_1 is the coefficient of the vertical dipole in ϕ_0 at a large distance from the origin. For the particular trapped mode given by the potential in (1), $p_1 = -2/K_0 \neq 0$ and so the heave potential does not exist at the trapped mode frequency for the pair of bodies illustrated in figure 2.

Existence of the diffraction potential

Once the incident wave has been subtracted out from the diffraction potential the remaining scattered potential satisfies

$$\frac{\partial \phi_d}{\partial n} = -\frac{\partial}{\partial n}[e^{iK_0 x - K_0 y}] \tag{5}$$

on the bodies, assuming that the wave is incident from large negative x. Under the assumption that the diffraction potential exists an application of Green's theorem to ϕ_0 and ϕ_d in the fluid region outside the bodies gives

$$\int_{\text{bodies}} \phi_0 \frac{\partial}{\partial n}[e^{iK_0 x - K_0 y}] \, dS = 0, \tag{6}$$

which is a necessary condition for the existence of ϕ_d. However, an application of Green's theorem to ϕ_0 and $e^{iK_0 x - K_0 y}$ shows that (6) is true when ϕ_0 is *any* trapped mode potential. Under the assumption that the Fredholm alternative applies this is sufficient for the diffraction potential to exist.

Resonance in a specific initial value problem

In order to investigate the physical significance of the non-existence of a steady-state potential, an initial value problem in which the fluid is given an oscillatory pressure forcing on the free surface is investigated. Let $\Phi_p(x,y,t)$ satisfy Laplace's equation, have zero normal derivative on the bodies, decay to zero at large depths and satisfy the condition

$$\frac{\partial^2 \Phi_p}{\partial t^2} - g\frac{\partial \Phi_p}{\partial y} = \phi_0(x,0)\cos\omega_0 t, \quad t>0, \tag{7}$$

on the free surface outside the bodies with initial conditions

$$\Phi_p(x,0,0) = \frac{\partial \Phi_p}{\partial t}(x,0,0) = 0. \tag{8}$$

If a steady state solution to this problem were sought in the form $\Phi_p = Re[\phi_p(x,y)e^{-i\omega_0 t}]$, a boundary value problem for ϕ_p would be obtained which does not have a solution. (The necessary condition for the existence of $\phi_p(x,y)$ is $\int \phi_0^2(x,0)\,dx = 0$, where the integral is taken over the whole free surface outside the bodies. As ϕ_0 is real and not equal to zero everywhere on $y=0$, this condition is not satisfied.) However, from the equations and boundary conditions satisfied by the trapped mode potential ϕ_0 it is straightforward to show that the solution to the initial value problem is given by

$$\Phi_p = \frac{t}{2\omega_0}\phi_0(x,y)\sin\omega_0 t. \tag{9}$$

Clearly the amplitude of the oscillation of the resulting fluid motion grows with time and resonance occurs. Physically the problem may be interpreted as a situation in which energy is continually fed into a local oscillation and is not carried away from the vicinity of the body through any wave motion.

Conclusion

An example has been given which shows that resonance can occur in the two-dimensional, linear, water wave problem in an unbounded region when there is at least one trapped mode solution of the related frequency domain problem. The more general initial value problem in which the free surface condition (7) is replaced by

$$\frac{\partial^2 \Phi_p}{\partial t^2} - g\frac{\partial \Phi_p}{\partial y} = Re[f(x)\,e^{-i\omega_0 t}], \quad t>0, \tag{10}$$

where $f(x)$ is square integrable, is currently under investigation. In particular, it is hoped to extend the results of Goldstein (1969) for waveguides and expand $f(x)$ as an integral over all scattering potentials plus a sum over all trapped mode potentials and then derive an explicit expression for Φ_p with the use of a Laplace transform in time.

References

Goldstein, C. I. 1969 'Eigenfunction expansions associated with the Laplacian for certain domains with infinite boundaries. I' *Transactions American Mathematical Society*, Vol. 135, pp 1 - 31.

McIver, M. 1996 'An example of non-uniqueness in the two-dimensional linear water wave problem.' *J. Fluid Mechanics*, Vol. 315, pp 257 - 266.

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DISCUSSION

Evans D.V.: What are the implications of your non-uniqueness on, say, the heave added mass of a catamaran having hull cross-section precisely of the form which gives your non-uniqueness?

McIver M.: The heave added mass doesn't actually exist at the exact frequency at which non-uniqueness occurs. Although I haven't done the computations, in the corresponding case for a body in a channel, the added mass tends to plus or minus infinity either side of the trapped mode frequency. I would expect the same thing to happen in this case.

Schultz W.: To apply the Fredholm alternative theorem as you have described requires the solution to be self-adjoint. What new insights could be found if this constraint were dropped? Could asymmetric solutions be found?

McIver M.: I agree that the operator in this case is self-adjoint. However, I don't think that you need to have a non-self-adjoint operator in order to obtain an asymmetric configuration of bodies. In the example I presented you could construct 2 bodies of different size by taking parts of different streamlines.

Tuck E.O.: If non-uniqueness occurs at some particular wavenumber $k = k_o$, this means that output quantities like added mass do not exist at $k = k_o$. An interesting question is what then happens at $k = k_o \pm \varepsilon$ where ε is arbitarily small. Can one make the output as large as one likes by choosing ε sufficiently small? That is, on a graph, does the output appear to "go to infinity" as $k \to k_o$?

McIver M.: I think this is correct. Certainly when you get trapped modes about a cylinder in a channel the added mass goes to plus or minus infinity either side of the trapped mode frequency.