THE HYDRODYNAMIC LOAD AT THE INTERSECTION OF A CYLINDER WITH THE WATER SURFACE

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1. Background

The evaluation of hydrodynamic forces on cylinders is important in the design of offshore structures against wave loads. Where the cylinder diameter is large enough for the problem to be one of potential flow, the standard method of finding the wave loads is by Stokes' expansion computations, which generate "1st order" loads proportional to wave height, "2nd order" loads proportional to (wave height)^2, etc. In early 1992, however, the latest generation of Norwegian offshore structures were found during scale-model trials to suffer from severe high-frequency "ringing" loads in extremely high waves, which had not been predicted by such computations, of either 1st or 2nd order.

It has recently been recognised that "ringing" loads may in principle be impossible to compute by Stokes' expansion, because the waves responsible are so high in relation to the cylinder diameter that Stokes' expansion may actually *diverge* (Rainey, 1995 - it is of course also known that Stokes' expansion diverges without any cylinder interaction, if the waves are sufficiently steep, see Schwartz, 1974). Instead it may be necessary to use an altogether different approach. One such is the "slender body theory" introduced in Rainey (1989), in which the Stokes' expansion parameter of (waveheight/wavelength) is replaced by the slenderness parameter ([cylinder diameter]/[wavelength, cylinder length etc.]), and fully non-linear incident waves are included. Another is the perturbation scheme introduced in Faltinsen, Newman, & Vinje (1995), in which variables are expanded about their values on a horizontal plane moving up and down with the 1st (Stokes') order incident wave. These latter authors consider an infinite fixed vertical slender circular cylinder in Stokes 3rd order waves, and obtain the appropriate special case of Rainey's (1989) result, except that Rainey's point load at the water surface intersection is replaced by one eight times larger. As a result, the total 3rd harmonic load on the cylinder is almost doubled, which suggests that the results in Rainey (1989) could be misleading for predictions of "ringing".

2. Jefferys' elementary energy arguments

In Rainey (1989) the water surface is assumed to be constrained by a "wavy lid" in the shape of the incident waves, so that it does not distort around the cylinder. The point load at the water surface intersection is found from Lagrange's equations to have two components, both normal to the cylinder, with one parallel to the water surface and one in a plane normal to it. When a circular cylinder is moving broadside at constant velocity u in a such a plane, and enters still water obliquely, only the latter force acts. Its value is $\frac{1}{2}\rho c u^2 \tan\alpha$, where ρ is the water density, c is the cylinder cross-sectional area, and α is the angle of the cylinder axis to the vertical.

As pointed out in Rainey (1989), this is clearly correct from the elementary consideration that the rate of increase of wetted length is $u\tan\alpha$, and the kinetic energy per unit wetted length is $\frac{1}{2}\rho c u^2$, and no other hydrodynamic forces are doing any work (because the cylinder is moving broadside, i.e. with zero axial velocity. The loads at the immersed end, also given in Rainey 1989, then do no work). It has subsequently been pointed out by Jefferys (see Rainey, 1995) that this argument is *still valid* when the surface is allowed to distort, because the energy in the surface distortion is *constant*.

Jefferys has recently extended this argument (Jefferys, 1994) and applied it to a fixed vertical cylinder in long waves, by adopting a frame of reference moving with the water. Rainey's point load expression above then becomes $\frac{1}{2}\rho c v^2 \dot{v}/g$ because $\tan\alpha$ is $\dot{v}/g$ where v is the horizontal water velocity and g is the acceleration due to gravity. The eightfold increase in this load predicted by Faltinsen, Newman & Vinje therefore implies an additional force of $(7/2)\rho c v^2 \dot{v}/g$. Since the key extra feature in these authors' analysis is surface distortion, Jefferys then makes the radical suggestion that this additional force is due to *the rate-of-change of the energy stored in the surface distortion*, which is a quite different mechanism from Rainey's original force. Jefferys then deduces that if this is so, the energy stored in the distortion must be the integral of the force's rate-of-working $(7/2)\rho c v^3 \dot{v}/g$ in the moving reference frame, which comes to $(7/8)\rho c v^4/g$. This highly interesting conjecture is explored below.

3.The "distorted wavy lid" approximation

The Lagrange-equation argument in Rainey (1989) by no means requires that the shape of the water surface be that of the incident wave. This is merely the simplest choice: for more accurate answers an obvious suggestion made in Rainey (1995) is to use instead the zero-pressure surface in the "fully immersed" flow (i.e. in the flow around the structure produced by a suitably-continuous extrapolation of the incident wave above its normal surface). This new surface is then treated in exactly the same way as the "wavy lid" in Rainey (1989), i.e. the exact kinematic boundary condition is applied there, just as if it were a moving "lid".

As before, the application of this kinematic boundary condition will produce additional flows which will cause the pressure there to be non-zero: however, the effect will be less since the zero-pressure surface in the fully immersed flow is a much better approximation to the true distorted shape of the free surface.

Critically, this "distorted wavy lid" approximation, like the simpler "wavy lid" approximation, makes no assumptions about the incident waves, which can be fully-nonlinear and even breaking. This is an important advantage over the perturbation scheme introduced in Faltinsen, Newman, & Vinje, (1995), which requires the surface position of the first-order Stokes component of the wave - that position is of course a moot point when the wave is breaking!

4. Simplest case: partially-immersed vertical circular cylinder moving at constant velocity in still water
In this simplest case, the "fully immersed" velocity potential ϕ_F is just:

$$\phi_F = -\frac{ub^2}{r}\cos\theta \tag{1}$$

where u is the cylinder velocity, b is its radius, and r,θ are the usual cylindrical polar coordinates, with $\theta = 0$ in the direction of cylinder motion. The elevation η of the "distorted wavy lid" above the undisturbed water level follows immediately from the associated pressure field, thus:

$$\eta = \frac{u^2}{2g}[2(b/r)^2\cos 2\theta - (b/r)^4] \tag{2}$$

The kinematic boundary condition on this "lid" is simply that the normal fluid velocity equals the normal lid velocity: this leads to an incident potential ϕ_I satisfying:

$$\frac{\partial\phi_I}{\partial n} = \boldsymbol{v}.\nabla\eta \tag{3}$$

there, where $\boldsymbol{v}$ is the vector velocity of translation of the "lid" (viz. u in the direction $\theta = 0$). In addition there will be a diffracted potential ϕ_D satisfying $\partial\phi_D/\partial n = 0$ on the "lid" and:

$$\frac{\partial\phi_D}{\partial n} = -\frac{\partial\phi_I}{\partial n} \tag{4}$$

on the cylinder, and a lid-induced modification ϕ'_F to ϕ_F satisfying:

$$\frac{\partial\phi'_F}{\partial n} = -\frac{\partial\phi_F}{\partial n} = -\nabla\phi_F\,.\nabla\eta \tag{5}$$

on the "lid" and $\partial\phi'_F/\partial n = 0$ on the cylinder.

The fluid kinetic energy is simply:

$$\frac{1}{2}\rho\int_V \nabla[\phi_F+(\phi_I+\phi_D+\phi'_F)].\nabla[\phi_F+(\phi_I+\phi_D+\phi'_F)] \tag{6}$$

where ρ is the water density and V is its total volume up to the "lid". The group of potentials $(\phi_I+\phi_D+\phi'_F)$ is proportional to the Froude number[2] u^2/bg; if we assume this is small so that we can neglect terms proportional to $(u^2/bg)^2$ [in fact we have already assumed this in (3) and (5)], then we can write the kinetic energy as:

$$\frac{1}{2}\rho\int_V \nabla\phi_F\,.\nabla\phi_F + \rho\int_V \nabla\phi_F\,.\nabla(\phi_I+\phi_D+\phi'_F) \tag{7}$$

The second term can be evaluated by applying Green's theorem in the style of Rainey 1989 eqn. (5.4); taking advantage of the fact that $\partial(\phi_I+\phi_D+\phi'_F)/\partial n = 0$ on the cylinder, it becomes:

$$-\rho\int_S \phi_F\frac{\partial(\phi_I+\phi_D+\phi'_F)}{\partial n} = -\rho\int_S \phi_F\nabla\eta.(\boldsymbol{v}-\nabla\phi_F) \tag{8}$$

where S is the "lid" surface, or more conveniently, since we are ignoring terms of order $(u^2/bg)^2$, the horizontal plane. From the 2-D form of Gauss' theorem in this plane we note that:

$$0 = \int_C \phi_F \eta (\nu - \nabla\phi_F).\boldsymbol{n} = \int_S \nabla.[\phi_F \eta(\nu - \nabla\phi_F)] = \int_S \nabla\phi_F.\eta(\nu-\nabla\phi_F) + \int_S \phi_F \nabla\eta.(\nu - \nabla\phi_F) \qquad (9)$$

where C is the cylinder waterline (where $(\nu-\nabla\phi_F).n = 0$ by definition of ϕ_F) and we are taking advantage of the fact that $\nabla.(\nu-\nabla\phi_F) = \nabla.\nabla\phi_F = 0$. Thus we can re-write the energy (8) as:

$$\rho\int_S \eta\nabla\phi_F.(\nu-\nabla\phi_F) \qquad (10)$$

We can now readily add in the contributions from the first term in (7), and from the potential energy, to obtain the total energy in the surface distortion as:

$$\rho\int_S \eta[\nabla\phi_F.(\nu-\frac{1}{2}\nabla\phi_F) + \frac{g\eta}{2}] \qquad (11)$$

which is readily evaluated using (1) and (2) as:

$$\frac{\rho u^2}{2g}\int_b^\infty\int_0^{2\pi}[2(b/r)^2\cos 2\theta - (b/r)^4][u^2(b/r)^2(\cos 2\theta - \frac{1}{2}(b/r)^2) + \frac{u^2}{4}(2(b/r)^2\cos 2\theta - (b/r)^4)]r\,d\theta\,dr \qquad (12)$$

$$= \frac{\rho u^4}{2g}\int_b^\infty\int_0^{2\pi}[3(b/r)^4\cos^2 2\theta + \frac{3}{4}(b/r)^8]r\,d\theta\,dr = \frac{\rho\pi u^4}{g}\int_b^\infty[\frac{3}{2}(b/r)^4 + \frac{3}{4}(b/r)^8]r\,dr = \frac{\rho c u^4}{g}[\frac{3}{4}+\frac{3}{24}] = \frac{7\rho c u^4}{8g} \qquad (13)$$

which is precisely the value suggested by Jefferys.

5. Accelerating vertical cylinder: extra distortion energy, and forces on "lid"

This energy value has however assumed the simple constant-velocity "lid" shape (2). To complete Jefferys' argument it is necessary to check that this energy is the same when the cylinder is accelerating (and thus the "lid" has a new shape), and that the forces acting on the "lid" do negligible work.

The effect of a cylinder acceleration $\dot{u}$ on the above argument will be to add an additional term:

$$-\frac{\dot{\phi}_F}{g} = \frac{\dot{u}b^2}{gr}\cos\theta \qquad (14)$$

to the "lid" elevation (2). Also, the "lid" will have an additional vertical velocity because u is not constant; this can be written from (2) and (14) as:

$$\dot{\eta} = \frac{u\dot{u}}{g}[2(b/r)^2\cos 2\theta - (b/r)^4] + \frac{\ddot{u}b^2}{gr}\cos\theta \qquad (15)$$

where in both (14) and (15) the dot means the rate-of-change seen in an inertial frame moving instantaneously with the cylinder. This new velocity $\dot{\eta}$ (i.e. $-\dot{\eta}$ into the fluid) will add to the boundary condition (3) for ϕ_I, and thus lead via (8) to an additional term in the total energy expression (11), viz:

$$\rho\int_S \dot{\eta}\phi_F \qquad (16)$$

In the final integration (12) it may be seen that the only result of the extra term (14) is an extra potential energy $\rho/2g \int_S \dot{\phi}_F\dot{\phi}_F$, and the only result of the extra term (16) is an extra kinetic energy $-\rho/g \int_S \ddot{\phi}_F\phi_F$. The sum of their rates-of-change is $-\rho/g \int_S \dddot{\phi}_F\phi_F$, which can be evaluated as $-\rho Kc^2 \dddot{u} u/g$, where the constant K is proportional to $\log(L/b)$, and arises because the integral diverges logarithmically until the distance from the cylinder becomes comparable with its immersed length L (which is large, but finite, see Rainey 1989 Sect. 5). By Jeffery's argument, this implies a force on the cylinder of $-\rho Kc^2 \dddot{u}/g$. This is actually of first order in waveheight in Stokes' expansion, and vanishes in the slender-body limit by comparison with all the other forces, which are proportional to c, not Kc^2.

The forces acting on the "lid" are proportional to the extra pressures generated by the above group of potentials $(\phi_I + \phi_D + \phi'_F)$; these are at most proportional to the potentials themselves and thus to the surface distortion. The rate of working of these pressures is clearly also proportional to the surface distortion; thus the contribution to Jefferys' energy argument will be proportional to (distortion)^2. For the small distortion assumed above, this can be neglected compared with Jefferys' other terms, which are proportional to (distortion)^1.

6. Cross-check using Lagrange's equations

Jefferys' whole argument can be cross-checked by finding the hydrodynamic loads directly by the method in Rainey (1989), which is Lagrange's equations. To do this it is necessary to write the kinetic part of the energy in (13) in the form:

$$\frac{\rho c}{g}\left(\frac{v^3}{2} - Kc\bar{v}\right)u + \frac{1}{2}u\frac{\rho c v^2}{6g}u = Iu + \frac{1}{2}uMu = Iu + \frac{1}{2}Nu \qquad (17)$$

where we are treating the lid velocity v and the cylinder velocity u as separate variables - see Rainey (1989) eqn. (5.11). The associated force at the intersection (in the direction opposing motion) is then $dI/dt + dN/dt$, see Rainey (1989) eqn. (6.8) which is:

$$\frac{\rho c 3v^2\dot{v}}{2g} - \frac{\rho K c^2 \bar{v}}{g} + \frac{\rho c(2v\dot{v}u+v^2\dot{u})}{6g} \qquad (18)$$

where the force $-\rho Kc^2\,\bar{v}/g$ is the same as the force $-\rho Kc^2\,\bar{u}/g$ identified above, and can be ignored for the same reasons. In addition, we have the force from the $\Delta e/\Delta X$ term in Rainey (1989) eqn. (6.8). Using the kinetic-energy part of (11), plus (16), to obtain the total kinetic energy e associated with the surface distortion, we can express this force (again in the direction opposing motion) as:

$$-\rho\int_S \frac{\partial\eta}{\partial x}[\nabla\phi_F\,.(v-\frac{1}{2}\nabla\phi_F)] \;-\; \rho\int_S \frac{\partial\dot{\eta}}{\partial x}\phi_F \qquad (19)$$

where $\partial/\partial x = \cos\theta\,\partial/\partial r - r^{-1}\sin\theta\,\partial/\partial\theta$. From (2), (14) and (12) it may be seen that only (14) makes a contribution to the first term in (19). From (15) and (1) it may be seen that only the $(b/r)^4$ term in (15) makes a contribution to the second term in (19). Thus (19) becomes:

$$\frac{\rho c v u\dot{v}}{2g} + \frac{\rho c v\dot{v}u}{g} \qquad (20)$$

Since $v = u$ we have in total from (18) and (20) an intersection force of:

$$\frac{7\rho c u^2\dot{u}}{2g} \qquad (21)$$

which is precisely the same as that produced by the Jefferys argument. It is interesting that the hydrostatic pressure, which has to be included as an extra forces with the Lagrange method (see Rainey 1989 eqn. (6.3)), gives zero result in the present example. This is readily seen by writing down the integral of $\rho g\eta^2$ around the cylinder - even when the contribution from (14) is included, the result is zero.

7.Discussion

The "slender body theory" in Rainey (1989) can evidently be extended as suggested in Rainey (1995), to include a force at surface intersections attributable to surface distortion. If this is done it gives the same results in the limiting case of small waves as those obtained by Faltinsen, Newman & Vinje (1995). This shows that the perturbation scheme adopted by these latter authors appears, for all its novelty, merely to give Stokes' expansion results, which could perhaps be obtained more easily by other means.

"Ringing" loads on offshore structures are, by contrast, caused by extremely steep waves, see Jefferys & Rainey (1994). An interesting case, featured in that paper, is a wave with the classical 120-degree sharp Stokes crest. Here the additional surface intersection force from surface distortion is likely to be less important relative to the existing intersection force in Rainey (1989). This can be seen by adopting a frame of reference moving with the wave crest: as a cylinder crosses the crest, there will be a sudden discontinuity in the intersection angle, but not in the relative fluid velocity or its rate-of-change. Thus the new force appears to be relatively less important in exciting "ringing", than it is in producing third-harmonic loads in small waves.

References

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DISCUSSION

Miloh, T.: Your approach uses just the 2-D added mass - should this not be corrected for the proximity of the water surface?

Rainey, R. C. T.: That is an interesting point, which I touched on in the discussion of my paper at the 1990 workshop. The effect you describe is of course the basis of Kelvin's original calculation of the interaction between a moving sphere and a fixed wall (Lamb's *Hydrodynamics*, 1932, Art. 137). I believe, however , that surface interactions of this type are of a higher order in slenderness than my other terms - an analogous example is the interaction between different cylinders.

Faltinsen, O.: You showed an illustration of a small T.L.P. where the leg diameter was only 6 m. Surely the drag loads, caused by flow separation, will then be large.

Rainey, R. C. T.: Indeed - although not perhaps as large as one might think, because small T.L.P.s have large first order horizontal motions, reducing the leg motions relative to the water. Since there is very little vertical motion of the T.L.P., however, there will clearly be severe flow separation around the pontoons - particularly in view of their flat, rectangular, cross-section.