Wave scattering by periodic arrays of breakwaters

R. Porter and D. V. Evans
School of Mathematics, University of Bristol, Bristol, BS8 1TW, UK

Introduction

A possible scheme for a breakwater is the use of an array of identical in-line barriers with gaps between them enabling vessels to pass through. The simplest model of such a breakwater design consists of a periodic array of thin-walled breakwaters in the form of identical vertical barriers extending throughout the water region. A number of authors have considered the problem of scattering of an incident wave by such an arrangement including, more recently, Dalrymple & Martin (1990) who consider normal incidence only, Williams & Crull (1993) who exploit a technique used by Achenbach & Li (1986).

The approach used here, whilst restricted to in-line barriers enjoys certain advantages over the methods used in the above literature. Appropriate eigenfunction expansions are used to obtain two singular integral equations, one for the jump in the pressure across a typical barrier, the other for the horizontal velocity through a typical gap between adjacent barriers as described in a related problem by Wu (1973). The important quantities are R_n and T_n, the reflection and transmission coefficients associated with different propagating modes, and measure the effectiveness of the breakwater. These coefficients are shown to be the solution of simple matrix equations which use integral properties of unknown functions that are the solution of a set of integral equations. For long enough waves only one reflected and one transmitted mode will exist and in this case it can be shown that any approximate solution produce upper and lower bounds to $|R_0|$ and $|T_0|$.

By choosing judicious series expansions of the unknowns to reflect their physical characteristics, extremely accurate complementary bounds to $|R_0|$ and $|T_0|$ can be produced with a minimum of effort. Typically the upper and lower bounds for $|R_0|$ have a relative error of $O(10^{-4})$ using just five terms in the series expansion.

Although complementary bounds can only be guaranteed for sufficiently long waves, the results for $|R_n|$ and $|T_n|$ in shorter waves, when more than a single wave is reflected and transmitted, are also extremely close using either the velocity or potential jump formulations.

Formulation and solution

Cartesian co-ordinates are chosen with the barriers occupying $x = 0$, $0 < z < h$ where h is the water depth, in a periodic in-line array, with gaps $2a$ and distance d between their centres. A wave is incident upon the array from $x > 0$ making an angle $\pi - \theta_0 \in [0, \frac{1}{2}\pi)$ with the positive x-direction. Because the barriers extend throughout the depth the depth dependence can be extracted out and the linearised velocity potential $\Phi(x, y, z, t)$ witten as

$$\Phi(x, y, z, t) = \mathrm{Re}\, \phi(x, y) \cosh k(h - z) \mathrm{e}^{\mathrm{i}\omega t}, \tag{1}$$

where time-harmonic motion of frequency $\omega/2\pi$ is assumed and k is the real positive root of $\omega^2 = gk \tanh kh$. Then $\phi(x, y)$ satisfies

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + k^2 \phi = 0, \quad \text{in the fluid,} \tag{2}$$

$$\frac{\partial \phi}{\partial x} = 0, \quad \text{on the barrier.} \tag{3}$$

The incident wave is of the form

$$\phi_i(x, y) = \mathrm{e}^{-\mathrm{i}(\beta_0 y + \alpha_0 x)}, \tag{4}$$

having wavelength $\lambda = 2\pi/k$ where we have written $\beta_0 = k \sin\theta_0$, $\alpha_0 = k\cos\theta_0$. Because of the periodicity of the barriers the field at a point $y + d$ differs from that at y by a factor $\mathrm{e}^{-\mathrm{i}\beta_0 d}$, being the change in phase of the incident wave. Thus we may write

$$\phi(x, y + md) = \mathrm{e}^{-\mathrm{i}m\beta_0 d}\phi(x, y), \quad m = 0, \pm1, \pm2, \ldots \tag{5}$$

and we need only consider a single strip such as $y \in [-\frac{1}{2}d, \frac{1}{2}d]$, $-\infty < x < \infty$ with (5) providing the extension of the solution to the whole plane.

The most general form for $\phi(x, y)$ which ensures that (5) is satisfied is

$$\phi(x, y) = \sum_{n=-\infty}^{\infty} A_n \mathrm{e}^{-\mathrm{i}(\beta_n y + \alpha_n x)}, \tag{6}$$

where

$$\beta_n = \beta_0 + \frac{2n\pi}{d}, \quad \text{and} \quad \alpha_n = (k^2 - \beta_n^2)^{1/2}, \tag{7}$$

to satisfy (2). By analogy with the definitions of α_0 and β_0 earlier we define $\beta_n = k\sin\theta_n$ so that $\alpha_n = k\cos\theta_n$ It is easily seen that there will be only one reflected or transmitted mode if the spacing $d < \lambda/2$. In general the number of reflected and transmitted modes will depend on the value of d/λ and the angle of incidence, θ_0 and for now we assume that values of n such that $-r \le n \le s$ will result in propagating modes. For $\beta_n > k$ we define $\gamma_n = (\beta_n^2 - k^2)^{1/2}$ ($= -\mathrm{i}\alpha_n$ for $\beta_n < k$) and we introduce the functions $\psi_n(y) = \mathrm{e}^{-\mathrm{i}\beta_n y}$ that are orthogonal over $[-\frac{1}{2}d, \frac{1}{2}d]$ We can now write, for $x > 0$

$$\phi(x, y) = \mathrm{e}^{\gamma_0 x}\psi_0(y) + \sum_{n=-\infty}^{\infty} A_n \mathrm{e}^{-\gamma_n x}\psi_n(y), \tag{8}$$

whilst for $x < 0$

$$\phi(x, y) = \sum_{n=-\infty}^{\infty} B_n \mathrm{e}^{\gamma_n x}\psi_n(y) \tag{9}$$

and we have assumed $\gamma_n = -\mathrm{i}\alpha_n$ for $n = -r, \ldots, s$, $\gamma_n > 0$ otherwise. Since A_n and B_n are the reflection and transmission coefficients for $n = -r, \ldots, s$, we let $R_n = A_n$ and $T_n = B_n$ for these n.

Continuity of $\phi_x|_{x=0}$ and $\phi|_{x=0}$ across the gap L_g and the condition of no-flow across the barriers now gives, after some algebra

$$\int_{L_g} u_n(t) K(y, t)\, \mathrm{d}t = \psi_n(y), \quad y \in L_g, \tag{10}$$

($K(y, t)$ known) and

$$\int_{L_g} u_n(y)\psi_m(y)\, \mathrm{d}y = S_{mn}, \tag{11}$$

where $\mathbf{S} = \{S_{mn}\}$, for $m, n = -r, \ldots, s$ and with

$$\begin{aligned} &\mathbf{R} = (R_{-r}, R_{-r+1}, \ldots, R_{s-1}, R_s)^{\mathrm{T}}, \\ &\mathbf{A} = \operatorname{diag}\{\alpha_n d\}_{n=-r,\ldots,s} \\ &\mathbf{B} = (0, \ldots, 0, 1, 0, \ldots, 0)^{\mathrm{T}}, \quad \text{entry at posn. corresponding to } n = 0 \end{aligned} \tag{12}$$

it turns out that
$$\mathbf{iA(R - B) = SR}, \tag{13}$$
giving
$$\mathbf{R} = -\mathrm{i}(\mathbf{S} - \mathrm{i}\mathbf{A})^{-1}\mathbf{AB}, \tag{14}$$
whilst
$$\mathbf{R} + \mathbf{T} = \mathbf{B}, \tag{15}$$
where
$$\mathbf{T} = (T_r, T_{-r+1}, \ldots, T_{s-1}, T_s)^{\mathrm{T}}. \tag{16}$$

A judicious expansion of the unknown $u_n(y)$ involving Chebychev polynomials weighted by the anticipated square-root singularity in $u_n(y)$ at the end-points results in a simple but accurate approximation to $\mathbf{S}$ in the form
$$\mathbf{S}^{\mathrm{T}} = \overline{\mathbf{F}}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{F} \tag{17}$$
where $F_{mn} = J_m(\beta_n a)$ and
$$K_{mn} = \sum_{r=-\infty}^{\infty}{}'(\gamma_r d)^{-1} J_m(\beta_r a) J_n(\beta_r a) \tag{18}$$
An alternative formulation in terms of the jump in ϕ across one of the barriers provides a set of integral equations similar to those in (10) with an associated matrix $\mathbf{P}$ as in (11) and eventually results in $\mathbf{R} = \mathrm{i}\mathbf{PA}(\mathbf{R} - \mathbf{B})$ from which it is clear that $\mathbf{P} = \mathbf{S}^{-1}$ whilst
$$\mathbf{R} = -\mathrm{i}(\mathbf{I} - \mathrm{i}\mathbf{P})^{-1}\mathbf{PAB}. \tag{19}$$
Here, a similar approximation to the set of unknown functions proportional to the jump in ϕ across a barrier gives rise to an approximation to $\mathbf{P}$ in the form
$$\mathbf{P}^{\mathrm{T}} = \overline{\mathbf{G}}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{G} \tag{20}$$
where $G_{mn} = (\beta_n d)^{-1} J_{m+1}(\beta_n c)$ and
$$M_{mn} = \sum_{r=-\infty}^{\infty}{}' \frac{(\gamma_r d)}{(\beta_r d)^2} J_{m+1}(\beta_r c) J_{n+1}(\beta_r c). \tag{21}$$
and where $c = \frac{1}{2}d - a$.

Results

Figures 1 and 2 show the variation of the reflection coefficients in two cases of normal and non-normal incidence. They show how only one reflected wave exists for long waves and how other modes occur at higher wavenumbers. Results obtained using both velocity and potential-difference formulations agree to within two significant figures in all cases and comparisons with the results of Dalrymple & Martin (1990) and Williams & Crull (1993) suggest a marked improvement.

References

[1] ACHENBACH, J. D. and LI, Z. L. (1986) Reflection and transmission of scalar waves by a periodic array of screens. *Wave motion*, **8**, No. 3, 225-234.

[2] DALRYMPLE, R. A. and MARTIN, P. A. (1990) Wave diffraction through offshore breakwaters. *J. Waterway, Port, Coastal & Oc. Eng.*, **116**, No. 6, 727-741.

[3] WILLIAMS, A.N. and CRULL, W.W. (1993) Wave diffraction by thin screen breakwaters. *J. Waterway, Port, Coastal & Oc. Eng.*, **119**, No. 6, 606-617.

[4] WU, C. P. (1973) Variational and iterative methods for waveguides and arrays. In *Computational Techniques for Electromagnetics*, (ed. R. Mittra), pp. 266-304. Pergamon.

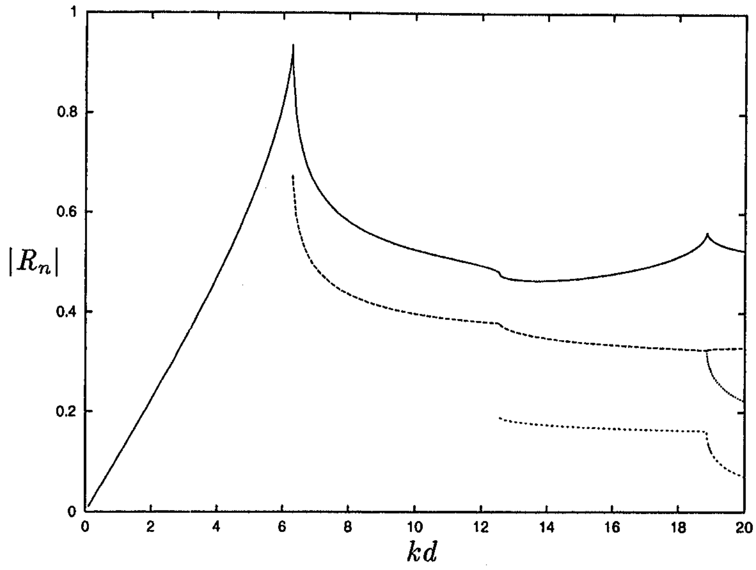

Figure 1: $|R_n|$ for $\theta = 0^o$ and $2a/d = 0.5$ (gap to screen size is 1:1).

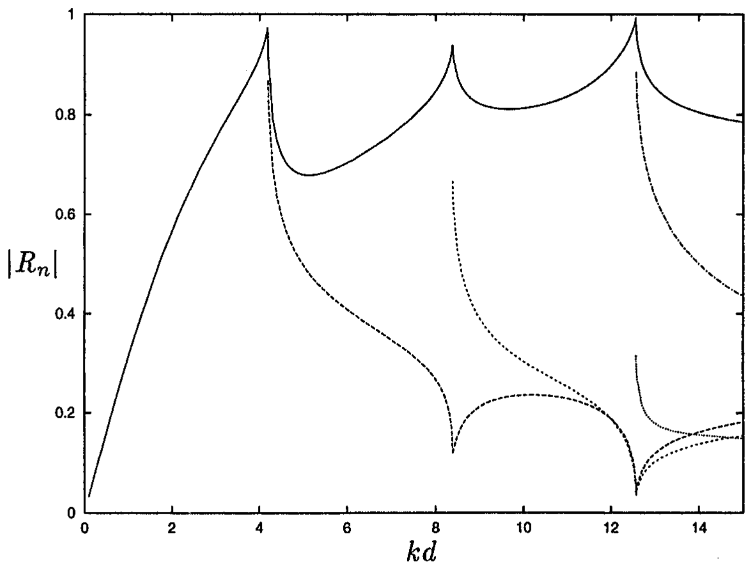

Figure 2: $|R_n|$ for $\theta = 30^o$ and $2a/d = 0.8$ (gap to screen size is 4:1).