The 2-D Neumann-Kelvin Problem for a Surface-piercing Tandem

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1. Introduction

A tandem of horizontal cylindrical bodies moves with constant velocity in the free surface of an inviscid, incompressible fluid under gravity. The resulting fluid motion is described by the linearized water-wave theory (the corresponding boundary value problem is usually referred to as the Neumann-Kelvin problem).

Almost exhaustive mathematical theory of the 2-D problem has been given by Kochin (1937) and Vainberg & Maz'ya (1973) for the case of totally submerged body. For a single semiimersed body this problem was treated both theoretically (Ursell 1981, Lenoir 1982, Kuznetsov & Maz'ya 1989) and numerically (see Suzuki 1982 and references cited therein). It was found that the problem is inconsistent, since it has a two-parameter set of solutions. Thus, the original statement, including Laplace's equation, the boundary conditions and the conditions at infinity, should be complemented by a couple of supplementary conditions. Some versions of such conditions were suggested in the above cited papers. Kuznetsov 1992 introduced a new pair of supplementary conditions, which provides that the resistance is purely wave-making, i.e. there are no splashes at bow and stern, and it is expressed by just the same formula as for totally submerged cylinder. In the present work we extend these supplementary conditions to the case of surface-piercing tandem.

2. Statement of the problem

The geometrical notations are given in figure 1:

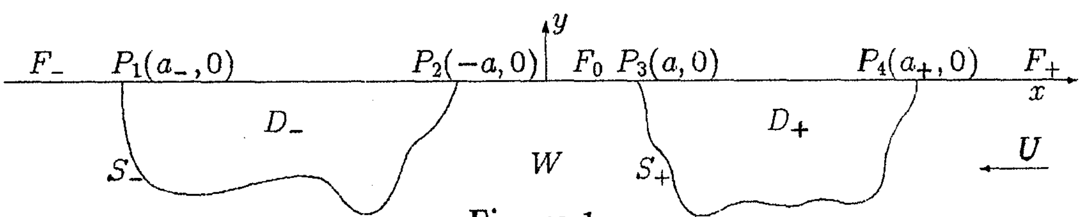

Figure 1 .

We assume that the fluid motion described by a velocity potential is steady-state in the coordinate system attached to the tandem. The potential must satisfy the following boundary value problem

$$\nabla^2 u = 0 \quad \text{in} \quad W, \qquad u_{xx} + \nu u_y = 0 \quad \text{on} \quad F = F_- \cup F_0 \cup F_+, \tag{1}$$

$$\partial u / \partial n = U \cos(n, x) \quad \text{on} \quad \operatorname{int} S = (S_- \cup S_+) \setminus \{P_1, P_2, P_3, P_4\}, \tag{2}$$

$$\lim_{x \to +\infty} |\nabla u| = 0, \qquad \sup\{|\nabla u| : (x, y) \in W \setminus E\} < \infty, \tag{3}$$

$$\int_{W \cap E} |\nabla u|^2 dx\, dy < \infty, \tag{4}$$

where U is the constant speed of the cylinders, $\nu = g\,U^{-2}$, g is the acceleration of gravity, $\mathbf{n}$ is the unit normal directed into W and E is an arbitrary compact set in $\overline{\mathbf{R}^2_-}$, containing $D_+ \cup D_-$ with contiguous parts of F.

The condition (4) allows to avoid strong singularities at P_{1-4}. This leads to existence of the derivatives $u_x(P_1), \ldots, u_x(P_4)$, which should be treated as the limits along the free surface.

To complete the statement in the case of tandem one has to add four conditions, instead of two supplementary conditions required for a single surface-piercing body. To introduce them in such a way, that the resistance is purely wave-making and its expression coincides with that obtained for totally submerged body, we need two auxiliary results. They hold for any solution to (1)–(4).

(i) The asymptotic representation at infinity can be written in the form (cp. with Kuznetsov & Maz'ya 1989):

$$u(x,y) = C + Q\log(\nu|z|) + H(-x)e^{\nu y}(\mathcal{A}\sin\nu x + \mathcal{B}\cos\nu x) + \psi(x,y), \quad |z| = |x+iy| \to \infty.$$

Here C is an arbitrary constant, H is the Heaviside function and the estimates $\psi = O(|z|^{-1})$, $|\nabla\psi| = O(|z|^{-2})$ are valid. To determine the constants $\mathcal{A}$, $\mathcal{B}$ and Q we have the equalities:

$$-\frac{\mathcal{A}}{2} = \int_S \left[u\frac{\partial}{\partial n}(e^{\nu y}\cos\nu x) - \frac{\partial u}{\partial n}e^{\nu y}\cos\nu x\right] ds + \sum_{\pm} \pm\left[\nu^{-1}u_x(x,0)\cos\nu x + u(x,0)\sin\nu x\right]_{x=\pm a}^{x=a\pm},$$

$$\frac{\mathcal{B}}{2} = \int_S \left[u\frac{\partial}{\partial n}(e^{\nu y}\sin\nu x) - \frac{\partial u}{\partial n}e^{\nu y}\sin\nu x\right] ds + \sum_{\pm} \pm\left[\nu^{-1}u_x(x,0)\sin\nu x - u(x,0)\cos\nu x\right]_{x=\pm a}^{x=a\pm},$$

$$\pi\nu Q + \sum_{\pm}\left[u_x(P_{3\pm1}) - u_x(P_{2\pm1})\right] = \nu\int_S \partial u/\partial n\, ds,$$

where $\sum_{\pm}$ means summation of two terms.

(ii) The formula for total resistance to forward motion looks as follows:

$$R = -\frac{\rho\nu}{4}(\mathcal{A}^2 + \mathcal{B}^2) - \frac{\rho}{2\nu}\left\{\left[u_x^2(x,0)\right]_{x=a-}^{x=-a} + \left[u_x^2(x,0)\right]_{x=+a}^{x=a+}\right\}$$

(ρ denotes a density of the fluid). One can derive it in the same way as in the case of single body (see Kuznetsov 1990). The first term in the right hand side gives the wave-making part of resistance, and it has the same form as for totally submerged body (cp. with Kochin 1937), but unlike the latter case the expressions for $\mathcal{A}$ and $\mathcal{B}$ contain out of integral terms. Another part of resistance (so called spray resistance) is connected with possible splashes at the bow and stern points of partially submerged bodies, and hence, with mass flow. In view of (2) the mean additional flux of fluid at infinity downstream due to the presence of cylinder $D_{\pm}$ is equal to

$$-2^{-1}\pi Q_{\pm} = (2\nu)^{-1}\left[u_x(P_{3\pm1}) - u_x(P_{2\pm1})\right].$$

The following pair of supplementary conditions

$$u_x(P_1) - u_x(P_2) = 0, \quad u_x(P_3) - u_x(P_4) = 0 \tag{5}$$

vanishes $Q_{\pm}$, and hence, the spray resistance is equal to zero. Due to (5) we can write $u_x^{\pm}$ instead of $u_x(P_{2\pm1}) = u_x(P_{3\pm1})$.

As the second pair of supplementary conditions we take the linear relations:

$$A_k^+u_x^+ + A_k^-u_x^- + B_k^+\Gamma_+ + B_k^-\Gamma_- + C_k\Gamma_0 = 0, \qquad k = 1,2, \tag{6}$$

where $\Gamma_{\pm} = u(P_{3\pm1}) - u(P_{2\pm1})$, $\Gamma_0 = u(P_3) - u(P_2)$ and the coefficients in (6) are defined as follows:

$$A^{\pm}_{(3\mp1)/2} = 2\nu^{-1}\sin(\nu l_{\pm}/2), \qquad A^{\pm}_{(3\pm1)/2} = 2\nu^{-1}\sin(\nu l_{\pm}/2)\cos\nu(a + L/2),$$
$$B^{\pm}_{(3\mp1)/2} = -\sin\nu(a + l_{\pm}/2)\sin\nu a - \cos(\nu L/2)\cos\nu(a + l_{\mp}/2),$$
$$B^{\pm}_{(3\pm1)/2} = -\cos(\nu l_{\pm}/2)\cos\nu(a + L/2), \quad C_{(3\pm1)/2} = \sin(\nu l_{\pm}/2)\sin\nu(a + L/2).$$

Here $l_\pm = |a_\pm \mp a|$ is the length of $D_\pm$ along the free surface, $L = l_+ + l_- + 2a$ is the length of tandem and only upper or lower signs should be taken both in subscripts and in superscripts.

The conditions (6) are degenerated when $l_+ + l_- = 2\nu^{-1}\pi n$, $2a = \nu^{-1}\pi(2m+1)$ or when $l_+ + 2a = 2\nu^{-1}\pi n$, $l_- + 2a = 2\nu^{-1}\pi m$ with positive integers n and m. Obviously, degeneration is impossible for sufficiently small values of ν. Under the assumption $u(P_1) + u(P_4) = 0$ (or, what is the same, under the appropriate fixation of constant term in the velocity potential) the non-degenerate conditions (5), (6) imply that terms out of integrals vanish in the formulae for $\mathcal{A}$ and $\mathcal{B}$. Conversely, if the out of integral terms are equal to zero, (5) holds and a, $l_\pm$ do not satisfy the above relations, then (6) is true provided $u(P_1) + u(P_4) = 0$.

3. On unique solvability of the problem (1)–(6).

To demonstrate that the formulation (1)–(6) is well-posed we seek a solution in the form

$$u(z) = \int_S \mu(\zeta) G(z,\zeta)\, ds_\zeta + \sum_{i=1}^{4} \mu_i G(z, a_i), \qquad (7)$$

where $G(z;\zeta)$ is Green's function (see e.g. Kuznetsov & Maz'ya 1989), μ is an unknown real density on int S and $\mu_1, \ldots, \mu_4$ are unknown real numbers. The function (7) satisfies (1) and (3)–(4). From (2) we get

$$-\mu(z) + 2\int_S \mu(z) \frac{\partial G}{\partial n_z}(z;\zeta)\, ds_\zeta + 2\sum_{i=1}^{4} \mu_i \frac{\partial G}{\partial n_z}(z, a_i) = 2f, \ z \in \text{int}\, S. \qquad (8)$$

This integral equation contains additional algebraic terms. Substituting (7) into (5), we obtain two algebraic equations with additional integral terms:

$$\int_S \mu(\zeta)\left[G_x(a_{k+1},\zeta) - G_x(a_k,\zeta)\right] ds + \sum_{i=1}^{4} \mu_i [G_x(a_{k+1}, a_i) - G_x(a_k, a_i)] = 0, \quad k = 1,3 \qquad (9)$$

Similarly, the supplementary conditions (6) yield two equations more of the same type as (9). We do not write them down explicitly, because it would take too much space.

Thus, we reduce our problem to the integro-algebraic system, containing five equations. The investigation of solvability and uniqueness properties for the integro-algebraic system and for the problem (1)–(6) follows the scheme proposed in Kuznetsov & Maz'ya 1989. On this way one arrives at the following result.

Theorem *Problem (1)–(6) has unique solution for all $\nu > 0$ except possibly a discrete sequence of values.*

4. Numerical calculations

A selection of results are presented in figure 2 for tandem consisting of two half-ellipses (ratio of horizontal and vertical half-axes is equal to 2). The Froude number is defined as $\mathrm{Fr} = (\nu L)^{-1/2}$. Its range in fig. 2(a,c) is chosen to avoid degeneration of the algebraic system. This system arises through discretisation of the Green formula integral equation complemented by (5) and (6).

Half-ellipses are equal ($l = l_+ = l_-$) in fig. 2(a) and $a/l = 1/3$ (1), $2/3$ (2), 1 (3). In fig. 2(b) the length $l_+(l_-)$ changes, all other lengths are fixed and $a/l_0 = 1/3$, where $l_0 = l_-(l_+)$. Also, $U/\sqrt{gl_0} = 1.5$ (1), 2.0 (2). Asymmetric tandems are considered in fig. 2(c,d): $l_- = l_0 = l_+/2$ (I) and $l_+ = l_0 = l_-/2$ (II). Also, $a/l_0 = 1/3$ (1), $2/3$ (2) in fig. 2(c) and Fr = 0.8 (1), 0.85 (2) in fig. 2(d). It is interesting to note intersection of curves in the cases (1) in fig. 2(c) and (2) in fig. 2(d).

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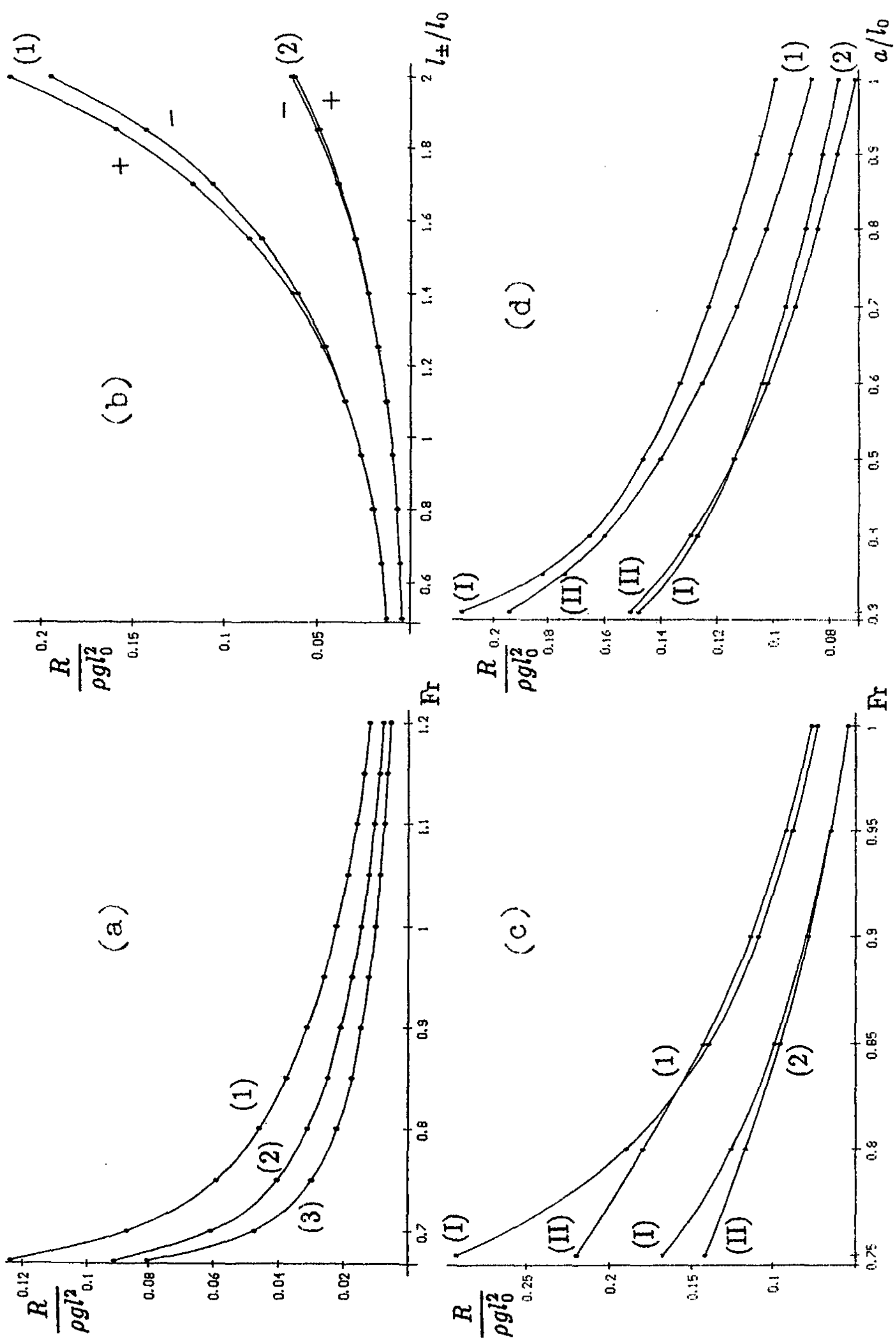

Figure 2.

DISCUSSION

Yeung, R. W.: For resistance as a function of Froude number as shown in Fig. 2, one would expect an oscillatory dependence because of interference effects. This does not appear to be the case. Can you comment?

Motygin, O. & Kuznetsov, N.: Numerical calculations of wave resistance have been performed only for a restricted range of the Froude number. This was caused by two circumstances.

1) The integral equation based on the Green formula, which was applied for the calculations, is inconsistent for a sequence of irregular Froude number values.

2) We had to avoid these irregular values, and our computer is not powerful enough for using more sophisticated methods without irregular values.