The existence or otherwise of trapped modes in channels

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Introduction

The existence of localized fluid oscillations in the neighbourhood of a body which is placed symmetrically about the centreline of a channel and extends uniformly throughout its depth was proved by Evans et al (1994). These 'trapped modes' are antisymmetric about the centreline of the channel and have finite energy. They occur at frequencies at which waves, antisymmetric about the centreline, are unable to propagate down the channel, i.e. below the cut-off frequency for the channel. At least one trapped mode is known to exist for any given body geometry but the numerical evidence of Evans & Linton (1991) indicates that the number of modes increases as the length of the body increases.

This work extends the recent work of McIver & Linton (1995) and proves that trapped modes do not occur in certain classes of channels which have variable cross-section. In addition, lower bounds for the trapped mode frequencies associated with channels which contain a body are obtained. The proof depends on a vector identity which is derived in the next section.

Theoretical analysis

Cartesian axes are chosen so that the xy−plane is in the undisturbed free surface and the z−axis points vertically upwards. The channel walls are at $y = \pm d(x)$ where $d(x) \to d$, a constant, as $|x| \to \infty$. Any body in the channel is assumed to be symmetrically placed about its centreline and to extend uniformly throughout the depth. Under the assumption of linear theory, the motion is described by a velocity potential

$$\Phi = Re[\phi(x, y) \cosh k(z + h)e^{-i\omega t}] \tag{1}$$

where h is the depth of the fluid, ω is the frequency of oscillation and k satisfies the dispersion relation $\omega^2 = gk \tanh kh$. A localized oscillation which is antisymmetric about the centerline of the channel is sought and so ϕ satisfies

$$(\nabla^2 + k^2)\phi = 0 \quad \text{in the fluid region in } y \geq 0 \tag{2}$$

with boundary conditions

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on the body and upper channel wall,} \tag{3}$$

$$\phi = 0 \quad \text{on } y = 0, \text{ outside the body} \tag{4}$$

and

$$\phi \to 0 \quad \text{as } |x| \to \infty, \ 0 \leq y \leq d(x). \tag{5}$$

A trapped mode corresponds to a nonzero solution of (2) - (6). Without loss of generality ϕ may be assumed to be real as the real and imaginary parts of ϕ must separately satisfy the boundary value problem.

The function v is defined by

$$v = \frac{\phi}{w} \tag{6}$$

where w is a strictly positive function. Elementary manipulation shows that v satisfies

$$\nabla^2 v + \frac{2}{w}\nabla w.\nabla v = -\frac{\phi}{w^2}(\nabla^2 w + k^2 w), \tag{7}$$

where (2) has been used. Further manipulation yields the identity

$$\nabla.(w^2 v \nabla v) = w^2 (\nabla v)^2 - \frac{\phi^2}{w}(\nabla^2 w + k^2 w). \tag{8}$$

Integration of (8) over a fluid region D with boundary ∂D with the use of the divergence theorem yields the result

$$\int_D w^2(\nabla v)^2 - \frac{\phi^2}{w}(\nabla^2 w + k^2 w)\, dV = \int_{\partial D} \phi\frac{\partial \phi}{\partial n} - \frac{\phi^2}{w}\frac{\partial w}{\partial n}\, ds \tag{9}$$

where $\partial/\partial n$ is a derivative in the outward normal direction to D. If, in addition to being strictly positive, w satisfies

$$(\nabla^2 + k^2)w \leq 0 \quad \text{in}\, D \tag{10}$$

then the LHS of (9) is non-negative. If the RHS of (9) can be shown to be non-positive then both sides must vanish identically and a examination of the integrand on the LHS shows that this can only be true if $(\nabla v)^2 \equiv 0$ throughout D, i.e. v is a constant. But $v = \phi/w$ and so this would imply that $\phi = Aw$ for some constant A. As $w > 0$, if $\phi = 0$ at any point on ∂D then $A = 0$ and hence $\phi \equiv 0$.

In the following sections, suitable functions w will be chosen to prove the nonexistence of trapped modes in a variety of circumstances or to obtain bounds for the trapped mode frequencies.

Channels with variable cross-section

Suppose that there is no body in the channel. Let D be the whole of the fluid region in $y \geq 0$ contained within the lines $x = \pm L$ and choose

$$w = \sin k(y + \epsilon) \tag{11}$$

where $\epsilon > 0$ is fixed but arbitrary. Clearly, $w > 0$ provided that $k(d(x)+\epsilon) < \pi$. Condition (10) is identically satisfied and so the LHS of (9) is non-negative. If L is allowed to tend to infinity then the RHS of (9) reduces to

$$-\int_{y=d(x)} \phi^2 k \cot k(d(x) + \epsilon)\, n_y\, ds$$

where n_y is the component of the outward normal to the channel wall in the y direction. This is non-positive provided that $n_y \geq 0$ on the upper channel wall and $k(d(x)+\epsilon) \leq \pi/2$. As ϵ can be made arbitrarily small, this latter condition is equivalent to $k < \pi/2d_{\max}$

where $2d_{\max}$ is the maximum width of the channel. Thus, no trapped modes exist for $k < \pi/2d_{\max}$. The cut-off frequency for a channel which has a width $2d$ at infinity is given by $\pi/2d$. If this is also the maximum width of the channel then no trapped modes exist below the cut-off.

Examples

1. There are no trapped modes below the cut-off frequency in a straight-walled channel, with a rectangular protrusion, $y = b$, $-a \le x \le a$, $b < d$.
2. Any trapped modes below the cut-off frequency in a straight-walled channel with a rectangular indentation $y = b$, $-a \le x \le a$, $b > d$ must be at frequencies in the range $\pi/2b \le k < \pi/2d$. This agrees with the predictions of Evans & Linton (1991).

Doubly antisymmetric modes

We now consider a channel with an indentation which is symmetric about $x = 0$ and is contained between $x = \pm a$ and seek trapped modes which are antisymmetric about the line $x = 0$ as well as the line $y = 0$. Thus, only the region $x \ge 0$, $y \ge 0$ is considered as the potential satisfies

$$\phi = 0 \quad \text{on}\, x = 0. \tag{12}$$

Let D be the fluid region in $x \ge a$ and choose $w = \sin k(y + \epsilon)$ as in (11). Equation (9) yields

$$\int_{x=a} \phi \frac{\partial \phi}{\partial x}\, dy = -\int_D w^2 (\nabla v)^2\, dV - \int_{y=d} \phi^2 k \cot k(d+\epsilon)\, dx \le 0, \tag{13}$$

when $kd < \pi/2$. Now let D be the fluid region contained between the lines $x = 0$ and $x = a$ and choose

$$w = \sin k(x + \epsilon), \; ka < \pi/2. \tag{14}$$

Equation (9) reduces to

$$\int_D w^2 (\nabla v)^2\, dV = \int_{x=a} \left(\phi\frac{\partial \phi}{\partial x} - \phi^2 k \cot k(a+\epsilon)\right) dy - \int_{y=d(x)} \phi^2 k \cot k(x+\epsilon)\, n_x\, ds \tag{15}$$

where n_x is the component of the outward normal in the x direction along the indentation. The LHS of (15) is clearly non-negative and, from (13) and (14), the RHS is non-positive if $ka < \pi/2$, $kd < \pi/2$ and $n_x \ge 0$ along the indentation in $x \ge 0$. This yields $\phi \equiv 0$ in the region $0 \le x \le a$. Thus, the LHS of (13) is zero and an examination of the RHS of (13) yields $\phi \equiv 0$ in $x \ge a$. Thus, if $a/d < 1$, there are no doubly antisymmetric modes below the cut-off frequency.

Example

There are no doubly antisymmetric trapped modes in a straight-walled channel with a rectangular indentation $y = b$, $-a \le x \le a$, $b > d$ if $a < d$, which agrees with the numerical predictions of Evans & Linton (1991).

Bounds for the lowest trapped mode frequency

We now consider a straight-walled guide containing a body symmetrically placed about its centreline. Let D be the total fluid region in $y \ge 0$ and choose

$$w = \cos ky, \; kd < \pi/2. \tag{16}$$

Equation (9) reduces to

$$\int_D w^2(\nabla v)^2\,dV = \int_{\text{body}} \phi^2 k \tan ky\, n_y\, ds + \int_{y=d} \phi^2 k \tan kd\, dx \tag{17}$$

where n_y is the component of the *inward* normal to the body in the y direction. If $n_y \le 0$ on that part of the body in $y \ge 0$ then (17) yields

$$\int_D w^2(\nabla v)^2\,dV \le k \sin kd \cos kd \int_{y=d} v^2\,dx, \tag{18}$$

using the fact that $v = \phi/w$. Following the method used by Simon & Ursell (1984), a series of parallel, straight lines are drawn from the line $y = 0$ to $y = d$ in the region $x > 0$ and another set in the region $x < 0$ which contain the body and are such that there is a line emanating from every point on $y = d$. As $v = 0$ on $y = 0$, integration along the line C_x joining $(x_0, 0)$ to (x, d) yields

$$v(x,d) = \int_{C_x} \frac{\partial v}{\partial s}\,ds = \frac{1}{\sin\beta}\int_{C_x} \frac{1}{w}\left(w\frac{\partial v}{\partial s}\right) dy \tag{19}$$

where β is the angle which the line makes with the x-axis, $(0 < \beta < \pi/2)$. Thus

$$v^2(x,d) \le \frac{1}{\sin^2\beta}\int_0^d \frac{1}{w^2}\,dy \int_{C_x} w^2\left(\frac{\partial v}{\partial s}\right)^2 dy = \frac{\tan kd}{k\sin^2\beta}\int_{C_x} w^2\left(\frac{\partial v}{\partial s}\right)^2 dy \tag{20}$$

using (16). Substitution of (20) into (18) gives, after some manipulation

$$\int_D w^2(\nabla v)^2\,dV \le \frac{\sin^2 kd}{\sin^2\beta}\int_{D'} w^2\left(\frac{\partial v}{\partial s}\right)^2 dV \le \frac{\sin^2 kd}{\sin^2\beta}\int_D w^2(\nabla v)^2\,dV \tag{21}$$

where D' is the region swept out by the lines, $D' \subset D$. Clearly if $kd < \beta$, this is only possible if $\phi \equiv 0$. Thus, trapped modes can only occur below the cut-off frequency at values of k satisfying $\beta \le kd < \pi/2$.

Example

The largest possible value of β which produces suitable lines which contain a rectangular block of length $2a$ and width $2b$ is given by $\tan\beta = (d-b)/a$. Values of β are compared with the numerical results of Evans & Linton (1991) in the table below.

b/d	a/d	β	numerical value of kd
0.25	0.05	1.504	1.569
0.25	0.2	1.310	1.531
0.25	0.6	0.896	1.243
0.25	1.0	0.644	0.974

References

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