How to remove secularity in the solution of diffraction-radiation problem with small forward speed

Š. MALENICA
Institut Français du Pétrole, BP 311
92506 Rueil-Malmaison, France

The purpose of this note is to show how to eliminate the secular behavior of the $\varepsilon\tau$ order potential in the solution of wave diffraction-radiation by a body advancing with low forward speed.
We recall first some basic notions and definitions in the frequency domain.
Within the frame of potential theory, the boundary value problem (B.V.P.) of diffraction-radiation with small forward speed U is described by the Laplace equation in the fluid domain Ω, no flow condition on the fixed boundaries, appropriate radiation conditions for diffracted and radiated parts and the following condition on the mean free surface ($z = 0$) :

$$-\omega_e^2\phi + g\frac{\partial\phi}{\partial z} + 2iU\omega_e\frac{\partial\phi}{\partial x} - 2iU\omega_e\nabla_0\bar{\phi}\nabla_0\phi + Ui\omega_e\phi\frac{\partial^2\bar{\phi}}{\partial z^2} = 0$$

where $\omega_e = \omega_0 - k_0U\cos\beta$ is the encounter frequency and ω_0 is the fundamental frequency of incoming waves. The potential $\bar{\phi}$ is the stationary potential due to the interaction of the curent with the body.
The potential ϕ is decomposed into incident, diffracted and radiated parts :

$$\phi = \phi_I + \phi_D - i\omega_e\sum_{j=1}^{6}\xi_j\phi_{Rj}$$

with the diffracted part satisfying the following free surface and body conditions :

$$\{-\omega_e^2\phi_D + g\frac{\partial\phi_D}{\partial z} + 2iU\omega_e\frac{\partial\phi_D}{\partial x} - 2iU\omega_e\nabla_0\bar{\phi}\nabla_0(\phi_I+\phi_D) + Ui\omega_e(\phi_I+\phi_D)\frac{\partial^2\bar{\phi}}{\partial z^2} = 0\}_{z=0}$$

$$\{\frac{\partial\phi_D}{\partial n} = -\frac{\partial\phi_I}{\partial n}\}_{S_c}$$

and the radiated part satisfying :

$$\{-\omega_e^2\phi_{Rj} + g\frac{\partial\phi_{Rj}}{\partial z} + 2iU\omega_e\frac{\partial\phi_{Rj}}{\partial x} - 2iU\omega_e\nabla_0\bar{\phi}\nabla_0\phi_{Rj} + Ui\omega_e\phi_{Rj}\frac{\partial^2\bar{\phi}}{\partial z^2} = 0\}_{z=0}$$

$$\{\frac{\partial\phi_{Rj}}{\partial n} = n_j + \frac{iU}{\omega_e}m_j\}_{Sc}$$

where S_c is the body wetted surface.
Diffracted and radiated potentials satisfy also a radiation condition in the form [3] :

$$\phi \sim \frac{h(\theta)}{\sqrt{r}}\frac{\cosh k_1(\theta)(z+H)}{\cosh k_1(\theta)H}e^{ik_1(\theta)r(1+O(\tau^2))} + O(1/r)$$

with $k_1(\theta) = k_0[1 + 2\tau\frac{\partial k_0}{\partial\nu}(\cos\theta - \cos\beta)]$ and $\tau = U\omega_0/g$.
This problem is usually solved by introduction of a new perturbation with respect to τ either in the B.V.P. [2,6] or in the expression for the Green function [4,5,7]. In the former case we write :

$$\phi(x,y,z) = \varphi(x,y,z) + \tau\psi(x,y,z)$$

which is applied to both the diffracted and radiated parts. Unfortunately this perturbation is nonuniform i.e. the solution for the potential ψ is secular with the following behavior at large r :

$$\psi \sim [f(\theta) r\cos\theta + g(\theta) r]\frac{e^{ik_0 r}}{\sqrt{r}}$$

and so valid only in the vicinity of the body.

In order to remove this secularity we make now use of multiple scale analysis [1]. For the sake of simplicity we consider only the diffraction problem for a vertical cylinder in water of finite depth. By introducing two new variables $\gamma = \tau r$ and $\delta = \tau r\cos\theta$, the following perturbation series for ϕ_D potential is assumed:

$$\phi_D(r,z,\theta) = \varphi_D(\gamma,\delta,r,z,\theta) + \tau\psi_D(\gamma,\delta,r,z,\theta)$$

After some manipulations we obtain the following B.V.P. at the corresponding orders :

$O(1)$

$$\begin{aligned}
\{\Delta\varphi_D &= 0\}_{\mathbf{x}\in\Omega} \\
\{-\nu\varphi_D + \frac{\partial\varphi_D}{\partial z} &= 0\}_{z=0} \\
\{\frac{\partial\varphi_D}{\partial r} &= -\frac{\partial\phi_I}{\partial r}\}_{r=a}
\end{aligned}$$

$O(\tau)$

$$\begin{aligned}
\{\Delta\psi_D &= -2\frac{\partial^2\varphi_D}{\partial r\partial\gamma} - \frac{1}{r}\frac{\partial\varphi_D}{\partial\gamma} - 2\frac{\partial^2\varphi_D}{\partial\delta\partial x}\}_{\mathbf{x}\in\Omega} \\
\{-\nu\psi_D + \frac{\partial\psi_D}{\partial z} &= -2i\frac{\partial\varphi_D}{\partial x} - 2k_0\cos\beta\varphi_D + 2i\nabla_0\bar{\phi}\nabla_0(\varphi_I+\varphi_D) - i(\varphi_I+\varphi_D)\frac{\partial^2\bar{\phi}}{\partial z^2}\}_{z=0} \\
\{\frac{\partial\psi_D}{\partial r} &\doteq \frac{\partial\varphi_D}{\partial\gamma} + \cos\theta\frac{\partial\varphi_D}{\partial\delta}\}_{r=a}
\end{aligned}$$

with $\nu = \frac{\omega_0^2}{g} = k_0\tanh k_0 H = -k_n\tan k_n H$.

The potential φ_D is the standard linear diffraction potential which can be calculated in the classical way. In the case of vertical circular cylinders it is expressed by eigenfunction expansion. The most general solution can be written in the form :

$$\varphi_D = F(\gamma,\delta)\sum_{n=0}^{\infty}\varphi_n(r,z,\theta) = F(\gamma,\delta)\sum_{n=0}^{\infty} f_n(z) g_n(r,\theta)$$

with :

$n = 0$

$$f_0(z) = \frac{\cosh k_0(z+H)}{\cosh k_0 H}\quad,\quad g_0(r,\theta) = \sum_{m=-\infty}^{\infty}\beta_{m0}H_m(k_0 r)e^{im\theta}$$

$n > 0$

$$f_n(z) = \frac{\cos k_n(z+H)}{\cos k_n H}\quad,\quad g_n(r,\theta) = \sum_{m=-\infty}^{\infty}\beta_{mn}K_m(k_n r)e^{im\theta}$$

We divide now the ψ_D potential into three parts [2] $\psi_D = \psi_{1D} + \psi_{2D} + \psi_{3D}$:

$$\begin{aligned}
\{\Delta\psi_{1D} &= -2\frac{\partial^2\varphi_D}{\partial r\partial\gamma} - \frac{1}{r}\frac{\partial\varphi_D}{\partial\gamma} - 2\frac{\partial^2\varphi_D}{\partial\delta\partial x}\}_{\mathbf{x}\in\Omega} \\
\{-\nu\psi_{1D} + \frac{\partial\psi_{1D}}{\partial z} &= -2i\frac{\partial\varphi_D}{\partial x} - 2k_0\cos\beta\varphi_D\}_{z=0} \\
\{\frac{\partial\psi_{1D}}{\partial r} &= v(z,\theta)\}_{r=a}
\end{aligned}$$

$$\begin{aligned}\{\Delta\psi_{2D} &= 0\}_{\mathbf{x}\in\Omega}\\ \{-\nu\psi_{2D}+\frac{\partial\psi_{2D}}{\partial z} &= 0\}_{z=0}\\ \{\frac{\partial\psi_{2D}}{\partial r} &= -v(z,\theta)+\frac{\partial\varphi_D}{\partial\gamma}+\cos\theta\frac{\partial\varphi_D}{\partial\delta}\}_{r=a}\end{aligned}$$

$$\begin{aligned}\{\Delta\psi_{3D} &= 0\}_{\mathbf{x}\in\Omega}\\ \{-\nu\psi_{3D}+\frac{\partial\psi_{3D}}{\partial z} &= 2i\nabla_0\bar{\phi}\nabla_0(\varphi_I+\varphi_D)-i(\varphi_I+\varphi_D)\frac{\partial^2\bar{\phi}}{\partial z^2}\}_{z=0}\\ \{\frac{\partial\psi_{3D}}{\partial r} &= 0\}_{r=a}\end{aligned}$$

The potentials ψ_{2D} and ψ_{3D} are non secular and can be calculated by classical methods, see [6] for example. The secular behavior arises in the solution for ψ_{1D}. The general form of the particular solution for this potential can be written as :

$$\begin{aligned}\psi_{1D} &= F(\gamma,\delta)\{-2(i\frac{\partial}{\partial x}+k_0\cos\beta)\sum_{n=0}^{\infty}\frac{1}{k_n}\frac{\partial k_n}{\partial\nu}[z\frac{\partial\varphi_n}{\partial z}+r\frac{\partial\varphi_n}{\partial r}+H(-\nu\varphi_n+\frac{\partial\varphi_n}{\partial z})]\\ &\quad+(C_1r+C_2r\cos\theta)\sum_{n=0}^{\infty}\varphi_n\}\end{aligned}$$

The constants C_1 and C_2 will be chosen so that the secular behavior of the potential is eliminated. This leads to the following condition :

$$\lim_{r\to\infty}\{-\frac{2}{k_0}\frac{\partial k_0}{\partial\nu}(ir\cos\theta\frac{\partial^2\varphi_0}{\partial r^2}+k_0\cos\beta r\frac{\partial\varphi_0}{\partial r})+(C_1r+C_2r\cos\theta)\varphi_0\}=0$$

which combined with Sommerfeld radiation condition for φ_0 :

$$\lim_{r\to\infty}[\sqrt{k_0r}(\frac{\partial\varphi_0}{\partial r}-ik_0\varphi_0)]=0$$

gives :

$$C_1=2ik_0\frac{\partial k_0}{\partial\nu}\cos\beta\quad;\quad C_2=-2ik_0\frac{\partial k_0}{\partial\nu}$$

We assume now the function $F(\gamma,\delta)$ in the form $F(\gamma,\delta)=A(\gamma)B(\delta)$. The fact that the potential ψ_{1D} must satisfy Laplace equation in the fluid (in fact, Poisson equation now) results in the following differential equation :

$$(2\sum_{n=0}^{\infty}\frac{\partial\varphi_n}{\partial r}+\frac{1}{r}\sum_{n=0}^{\infty}\varphi_n)[\frac{\partial A(\gamma)}{\partial\gamma}+C_1A(\gamma)]B(\delta)+2(\sum_{n=0}^{\infty}\frac{\partial\varphi_n}{\partial x})[\frac{\partial B(\delta)}{\partial\delta}+C_2B(\delta)]A(\gamma)=0$$

which gives :

$$A(\gamma)=e^{-C_1\gamma}\quad;\quad B(\delta)=e^{-C_2\delta}$$

So the final solution for φ_D is :

$$\varphi_D=e^{2i\tau rk_0\partial k_0/\partial\nu(\cos\theta-\cos\beta)}\sum_{n=0}^{\infty}\varphi_n(r,z,\theta)$$

and for ψ_{1D} :

$$\begin{aligned}\psi_{1D} &= e^{2i\tau rk_0\partial k_0/\partial\nu(\cos\theta-\cos\beta)}\{-2(i\frac{\partial}{\partial x}+k_0\cos\beta)\sum_{n=0}^{\infty}\frac{1}{k_n}\frac{\partial k_n}{\partial\nu}[z\frac{\partial\varphi_n}{\partial z}+r\frac{\partial\varphi_n}{\partial r}+H(-\nu\varphi_n+\frac{\partial\varphi_n}{\partial z})]\\ &\quad+2ik_0\frac{\partial k_0}{\partial\nu}(\cos\beta-\cos\theta)r\sum_{n=0}^{\infty}\varphi_n\}\end{aligned}$$

This solution satisfies all conditions and is now free of secularity.
In the case of infinite water depth we can perform the same analysis to obtain :

$$\psi_{1D} \;=\; e^{2i\tau r\nu(\cos\theta-\cos\beta)}\{-\frac{2}{\nu}(i\frac{\partial}{\partial x}+\nu\cos\beta)(z\frac{\partial\varphi}{\partial z}+r\frac{\partial\varphi}{\partial r})+2i\nu(\cos\beta-\cos\theta)r\varphi\}$$

with $\varphi_D = e^{2i\tau r\nu(\cos\theta-\cos\beta)}\varphi$.
Some authors [4,5] define the parameter τ as a function of encounter frequency $\tau = U\omega_e/g$, and the present analysis, with some minor modifications, can be applied also to this case. The final solution for ψ_{1D} is then :

$$\psi_{1D} = e^{2ik_0\tau x\partial k_0/\partial\nu}\{-2i\frac{\partial}{\partial x}\sum_{n=0}^{\infty}\frac{1}{k_n}\frac{\partial k_n}{\partial\nu}[z\frac{\partial\varphi_n}{\partial z}+r\frac{\partial\varphi_n}{\partial r}+H(-\nu\varphi_n+\frac{\partial\varphi_n}{\partial z})]-2ik_0\frac{\partial k_0}{\partial\nu}x\sum_{n=0}^{\infty}\varphi_n\}$$

but now :

$$\tau = \frac{U\omega_e}{g} \quad ; \quad \nu = \frac{\omega_e^2}{g} = k_0\tanh k_0H = -k_n\tan k_nH$$

The two solutions are equivalent, but the first one is easier to implement in the case of bottom mounted vertical cylinder due to the fact that a simple analytic solution for φ_D can be obtained without including evanescent wave modes.
For the general case of a body of arbitrary shape similar analysis can be applied. For infinite water depth the solution presented here is directly valid but for finite depth it should be slightly modified. In fact, the particular solution for ψ_1 changes [2] but, since the nature of secularity is the same, the derivation of the equations is straightforward.

Acknowledgements

I wish to thank Dr. Chen X.B. from Bureau Veritas and Prof. Molin B. from ESIM for their helpful discussions and comments.

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