A second-order solution for the wedge entry with small deadrise angle

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Introduction

We consider the classical problem of the water entry at constant velocity of a wedge into a fluid domain initially at rest. We assume the fluid to be incompressible and the flow irrotational; we neglect surface tension and we take the pressure as constant along the free surface.

In previous papers ([1], [2]), we presented an asymptotic study of the water entry problem for a blunt body. A similar study was also performed independantly by Howison *et al.* [3].

The solution procedure relies on the method of matched asymptotic expansions. The small parameter is the deadrise angle (or here, for convenience, its tangent). The following asymptotic solutions are matched — see figure 1:

- an outer solution defined on a length scale equal to the wetted width, $\frac{\lambda Ut}{\tan\theta}$;
- an inner solution defined close to spray root on a length scale equal to Ut $\tan\theta$;
- a jet solution defined along the wedge on a length scale equal to the wetted width, $\frac{\lambda Ut}{\tan\theta}$, and on a thickness scale equal to the thickness of the inner jet, $\delta Ut \tan\theta$.

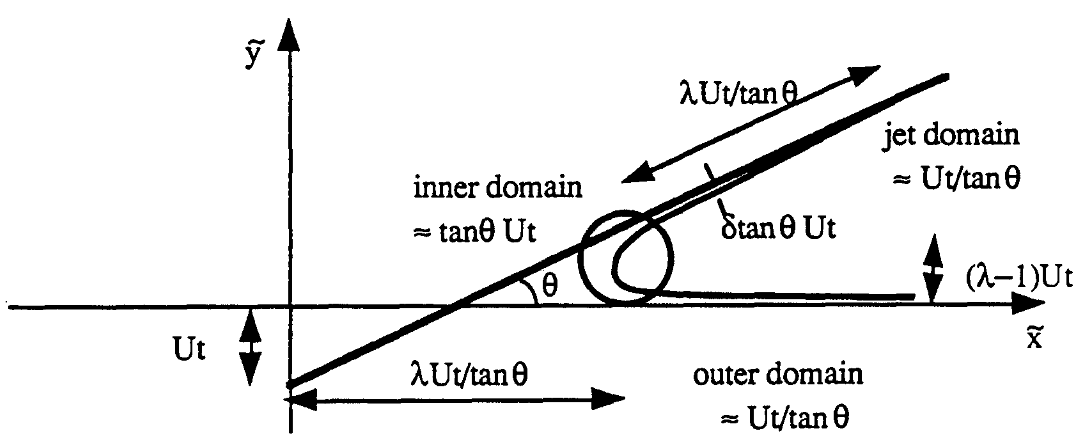


Figure 1 — Wegde entry: asymptotic solution for a small deadrise angle θ

Analytical asymptotic solutions can be found at the lowest order in each domain [2]. The outer solution corresponds to the flow around a flat plate in an unbounded fluid; the inner solution is given by a classical jet problem; and the jet solution leads to a rectilinear jet making a constant angle β with the wedge. The resulting composite solution satisfies the arc-length conservation property that results from the self-similarity of the flow. This study also led to an estimate of the behavior of the intersection angle between the free-surface and the wedge as the deadrise angle goes to zero, $\beta \approx \frac{(\tan\theta)^2}{2\pi}$.

This composite solution appears as a good approximation of the problem for small deadrise angle. However, it only statisfies volume conservation at order $\frac{(Ut)^2}{\tan\theta}$. The volume of the jet is of order $(Ut)^2$ and one has to solve the outer solution at the next order in order to account for the jet in the conservation of volume.

The objective of this paper is to provide a solution conserving volume at order $(Ut)^2$. For that purpose, we solve (numerically) the second-order outer solution and we show how volume conservation can be used to fully determine the solution of the problem. This seems to allow to overcome a difficulty that appeared [1] in the second-order problem due to the behavior of the solution close to the spray root.

Outer equations

The only length scales for the problem are Ut and $\frac{U^2}{g}$. When $\frac{gt}{U}$ is much smaller than 1 (i.e., for large impact velocity or small time), gravity can be neglected. Dimensional analysis enables one to conclude that the flow is self-similar. This means that the solution is independant of time in the self-similar coordinates.The outer problem is defined on a length scale equal to the wetted width $\frac{\lambda Ut}{\tan\theta}$. The spray root being the point $(\frac{\lambda Ut}{\tan\theta}, (1-\lambda)Ut)$ a change of variable is made to fix its position at the point (1,0). The new variables are defined by:

$$x = \frac{\tilde{x}\tan\theta}{\lambda Ut} \qquad y = \frac{\tilde{y}\tan\theta}{\lambda Ut} - \tan\theta\,(1-\frac{1}{\lambda}) \qquad \phi = \frac{\tilde{\phi}\tan\theta}{\lambda Ut} \qquad \eta = \frac{\tilde{\eta}\tan\theta}{\lambda Ut} - \tan\theta\,(1-\frac{1}{\lambda})$$

where the ~ variables are dimensionnal variables. Looking for a steady solution in term of the new variables, it can readily shown that the harmonic potential ϕ has to satisfy the free surface boundary conditions:

$$\phi - x\,\phi_x - \left[y + \tan\theta\,(1-\frac{1}{\lambda})\right]\phi_y + \frac{\tan\theta}{2\lambda}\nabla\phi\,\nabla\phi = 0 \quad \text{on } y = \eta(x)\,, \text{ for } x > 1$$

$$\eta + \tan\theta\,(1-\frac{1}{\lambda}) - x\,\eta_x + \frac{\tan\theta}{\lambda}(\eta_x\,\phi_x - \phi_y) = 0 \quad \text{on } y = \eta(x)\,, \text{ for } x > 1$$

as well the boundary condition on the wedge:

$$\phi_x \tan\theta - \phi_y = 1 \text{ for } 0 < x < 1\,,\ y = \tan\theta\,(x-1)$$

and on the symmetry condition:

$$\phi_x = 0 \quad \text{for } x = 0\,,\ y < -\tan\theta$$

First-order outer solution

Taking as expansions of the unknowns:

$$\phi = \phi_0 + o(1)\ ;\ \eta = \varepsilon\eta_1 + o(\varepsilon)\ ;\ \lambda = \lambda_0 + o(\varepsilon)\ ;\ \varepsilon = \tan\theta$$

the first order outer problem is easily found (see [2]).The harmonic potential ϕ_0 has to satisfy:

$$\phi_{0y} = -1 \quad \text{on } y = 0\,, \text{ for } 0 < x < 1$$
$$\phi_0 = 0 \quad \text{on } y = 0\,, \text{ for } x > 1$$
$$\phi_{0x} = 0 \quad \text{on } x = 0\,, \text{ for } y < 0$$

This is the classical problem of the flow around a flat plane and the complex potential is:

$$\phi_0 + i\psi_0 = -i\sqrt{(z^2-1)} + iz$$

The free surface elevation is given by the first order kinetic free surface condition:

$$(\eta_1-\eta_{1\infty}) - x\,(\eta_1-\eta_{1\infty})_x = \frac{\phi_{0y}}{\lambda_0} \quad \text{on } y=0\,,\ \text{for } x>1,$$

$$\text{with } \eta_{1\infty} = -\left(1-\frac{1}{\lambda_0}\right) \ \text{and } \eta_1-\eta_{1\infty}\to 0 \ \text{when } x\to\infty$$

The following result is obtain:

$$\eta_1-\eta_{1\infty} = \frac{1}{\lambda_0}\left[x \arcsin\frac{1}{x} - 1\right]$$

λ_0 can be determined by imposing volume conservation at first order.This leads to $\lambda_0=\pi/2$. For that volume of λ_0 the spray root (1,0) belongs to both the wedge and the free surface.However the solution is singular there since the velocity is infinite.

Second-order outer solution

Taking as expansions of the unknowns:

$$\phi = \phi_0 + \varepsilon\phi_1 + o(\varepsilon) \quad ; \quad \eta = \varepsilon\eta_1 + \varepsilon^2\eta_2 + o(\varepsilon^2) \quad ; \quad \lambda = \lambda_0 + \varepsilon\lambda_1 + o(\varepsilon^2)$$

The second order outer problem is found using a Taylor expansion for the wedge and the free-surface position.The harmonic potential ϕ_1 has to satisfy:

$$\phi_{1y} = \phi_{0x} - (x-1)\,\phi_{0yy} \quad \text{on } y=0\,,\ \text{for } 0<x<1$$

$$\phi_1 - x\,\phi_{1x} = x\,\eta_1\,\phi_{0xy} - \frac{1}{2\lambda_0}\phi_{0y}^2 + \frac{(\lambda_0-1)}{\lambda_0}\,\phi_{0y} \quad \text{on } y=0,\ \text{for } x>1$$

$$\phi_{1x} = 0 \quad \text{on } x=0\,,\ \text{for } y<0$$

This problem can be solved numerically. The second order free surface elevation has then to satisfy:

$$(\eta_2-\eta_{2\infty}) - x\,(\eta_2-\eta_{2\infty})_x = \frac{1}{\lambda_0}(\phi_{1y} + \eta_1\,\phi_{0yy}) - \frac{\lambda_1}{\lambda_0^2}\phi_{0y} \quad \text{on } y=0\,,\ \text{for } x>1 \text{ with } \eta_{2\infty} = -\frac{\lambda_1}{\lambda_0^2}$$

As λ_0 at first order, λ_1 is determined by imposing volume conservation at second order, taking into account the jet volume.

The volume conservation equation is:

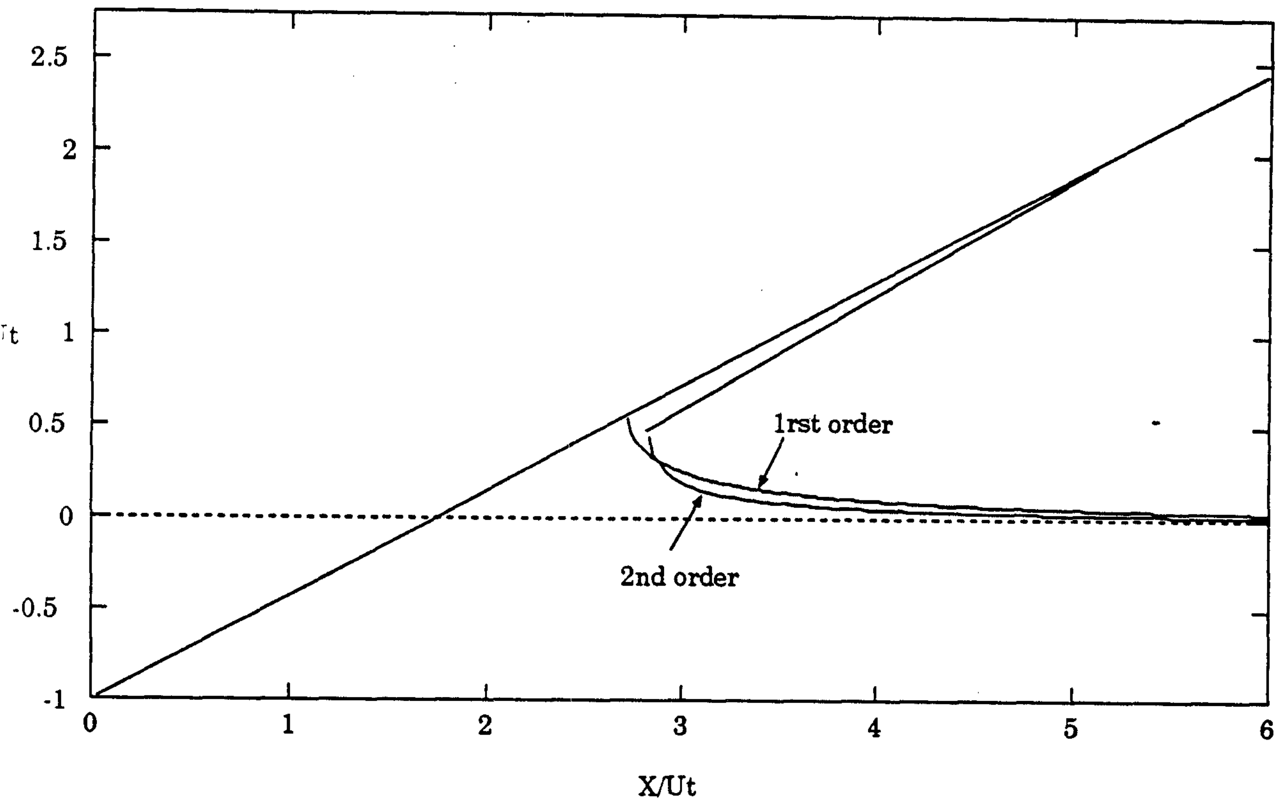
$$\frac{(Ut)^2}{2\tan\theta} = (\lambda-1)^2\,\frac{(Ut)^2}{2\tan\theta} + \int_{\frac{\lambda Ut}{\tan\theta}}^{\infty} \eta(x)\,dx + \delta\lambda\frac{(Ut)^2}{2}$$

Using the asymptotic expansion for η, and remembering ([2]) that the thickness of the jet is $\delta = \frac{\pi}{8\lambda}$ this equation reduces to:

$$\lambda_1 = -\lambda_0^2 \int_1^{\infty} \left[\eta_2 - \eta_{2\infty}\right] dx - \frac{\pi}{16}$$

The first term takes in account the second order free surface elevation while the second one is due to the jet volume. η_2 and λ_1 are computed in sequence until the iteration procedure converges. The results presented have been obtained with a deadrise angle of 30 degrees.

FIRST ORDER AND SECOND ORDER SOLUTIONS



References

[1] Cointe, R., and Armand, J.-L., 1987, "Hydrodynamic impact analysis of a cylinder," *ASME J. Offshore Mech. Arc. Engng*, Vol. 109, pp. 109-114.

[2] Cointe, R., 1991, "Free-surface flows close to a surface-piercing body," *Mathematical Approaches in Hydrodynamics*, SIAM, pp. 319-334.

[3] Howison, S.D., Ockendon, J.R., and Wilson, S.K., 1991, "Incompressible water-entry problems at small deadrise angle," *J. Fluid Mech.*, Vol. 222, pp. 215-230.

DISCUSSION

GREENHOW:
1) Does the second-order solution preserve arc length between Lagrangian particles on the free surface?
2) Does the contact angle versus deadrise angle satisfies the formula suggested by Johnstone and Mackie?
3) How are the forces and pressures affected by the second order solution?
FONTAINE & COINTE:
1) Yes, but only at first-order. Arc-length conservation at second-order would probably require next order solutions for the inner and jet solutions.
2) No. I already discussed that in the Workshop in Manchester and was puzzled at that time. Since then, I was told by Prof. Ockendon that Johnstone and Mackie result was found incorrect recently. Also, the recent fully nonlinear computations of Faltinsen and Zhao (reported in this Workshop) seems to agree quite well with our asymptotic estimate for very small deadrise angle.
3) We do not know because we did not compute them. Our previous experience with the problem for the circular cylinder (Cointe, R. & Armand, J.-L., "Hydrodynamic Impact Analysis of a Cylinder", ASME JOMAE, 1988) suggests that second-order effects are essential to explain the rapid decrease of the impact force with time (in $\sqrt{t}$). In the JOMAE paper, we used an approximate second-order solution for the outer flow. The present solution (that could easily be extended to the circular cylinder) would provide the basis for a more accurate computation of the second-order correction to the impact force in this case.

TUCK: It is not obvious from your figures that the results are better at 15° than they are at 30°. Should that not be so?
FONTAINE & COINTE: We believe that, as expected, the results are better at 15° than at 30°. Actually, the improvement gained by adding the second-order correction seems to disappear (and even to become negative) very rapidly as the deadrise angle increases — this is not too surprising for an asymptotic solution.

MARTIN: Please give a few more details on how you calculate the potential $\phi_1(x,y)$.
FONTAINE & COINTE: ϕ_1 is first computed on the linearized free surface by integrating the dynamic free surface condition, assuming that it vanishes at infinity. ϕ_{1n} is known on the wedge and is assumed to vanish at infinity. The potential ϕ_1 can then be obtained numerically by solving the integral equation. The numerical solver we use is the one we developed for the Sindbad code, as described in previous Workshops.