# SOME ASPECTS OF FREE SURFACE FLOW OVER A SUBMERGED BODY

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The steady two-dimensional free-surface flow of an incompressible, inviscid and irrotational stream of fluid which is bounded below by a rigid bottom and obstructed by a submerged body is considered. The submerged body is of Rankine type formed by a source and a sink of equal strengths $M$ and separated by a distance $S$. The body may be totally immersed in the fluid or could constitute a bump on the otherwise flat bottom but may not penetrate the free surface. A Cartesian-coordinate system $(X, Y)$ has its origin on the bottom of the stream at a distance $D$ below the source. Far upstream of the body the flow is uniform with speed $U$ and depth $H$; the restoring force in the negative $Y$-direction is gravity. These assumptions allow the introduction of a velocity potential $\Phi$ and stream function $\Psi$. The stream function is chosen to have the value 0 on the free surface and hence the value $-UH$ on the rigid bottom. The condition on the free surface where the pressure is uniform is obtained from Bernoulli's equation. Within the fluid the complex potential $W = \Phi + i\Psi$ is analytic.

The problem is non-dimensionalized using the transformations

$$z = x + iy = \frac{X + iY}{H}, \qquad w = \varphi + i\psi = \frac{\Phi + i\Psi}{UH}. \tag{1.1}$$

The geometry of the non-dimensional flow is shown in Fig. 1(a), where $d = D/H$, $s = S/H$, $m = M/UH$. The Bernoulli condition on the free surface can be written

$$\tfrac{1}{2}F^2q^2 + y_s = \tfrac{1}{2}F^2 + 1, \tag{1.2}$$

where $F = U/\surd(gH)$ is a Froude number, $q$ is the fluid speed and $y_S = y_S(x)$ is the free-surface elevation. The Rankine body formed by the source–sink combination is assumed to have a smooth profile, and a stagnation streamline will run from $-\infty$ to $+\infty$ passing through the points $S_\pm$ on the body.

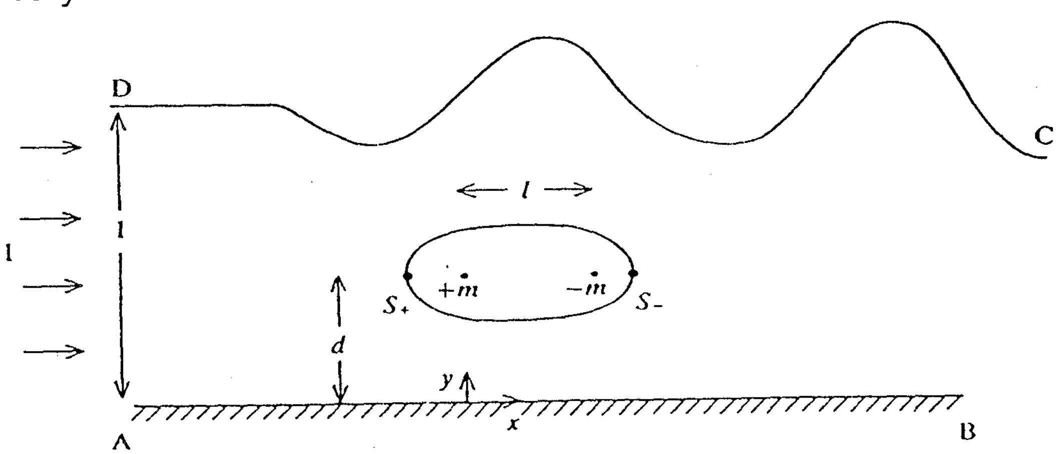


FIG. 1(a). The non-dimensional physical plane; $S_\pm$ are the front and rear stagnation points and $\pm m$ respresent the source and sink

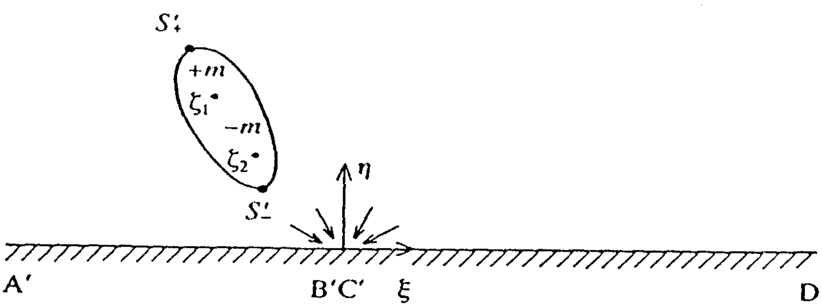


FIG. 1(b). The transform $\zeta$-plane and the corresponding points

It is convenient to transform the a-priori unknown region occupied by the fluid onto the upper half of the $\zeta$-plane where a complex potential may be found. The transformation used is of the type used by Bloor (1) and King and Bloor (2, 3). With the correspondence of the physical and transformed planes shown in Figs 1(a) and 1(b) the transformation takes the form

$$\frac{dz}{d\zeta} = -\frac{1}{\pi\zeta} \exp\left\{-\frac{1}{\pi}\int_{t=0}^{\infty} \frac{\theta(t)\, dt}{\zeta - t}\right\}, \tag{1.3}$$

where $\zeta = \xi + i\eta$ and $\theta(t)$ is the angle made by the tangent to the free surface with the $x$-axis at the point which corresponds to $\xi = t$. When $\xi$ is real and positive the integral in (1.3) becomes a principal value together with a contribution $i\theta(\xi)$ which ensures that $dz/d\zeta$ is an analytic function of $\zeta$. It is convenient to write

$$P = -\frac{1}{\pi} \int_{t=0}^{\infty} \frac{\theta(t)\, dt}{\xi - t},$$

where the bar through the integral sign denotes the Cauchy principal value of the integral. On the free surface $\xi > 0$,

$$\frac{dz}{d\xi} = -\frac{1}{\pi\xi} \exp\{P + i\theta\}. \tag{1.4}$$

In the $\zeta$-plane the flow is that of a sink of strength $1/\pi$ at the origin together with a source and sink of strength $m$ at the points $\zeta_1$ and $\zeta_2$, together with an image system, giving a complex potential

$$w = -\frac{1}{\pi} \log \zeta + m\{\log(\zeta - \zeta_1) + \log(\zeta - \bar{\zeta}_1) - \log(\zeta - \zeta_2) - \log(\zeta - \bar{\zeta}_2)\}. \tag{1.5}$$

The fluid velocity on the free surface is found from (2.4) and (2.5) to be

$$u - iv = qe^{-i\Theta} = \{1 - \varepsilon L(\xi, \zeta_1) + \varepsilon L(\xi, \zeta_2)\} \exp\{-P - i\theta\}, \tag{1.6}$$

where $\Theta$ is the angle made by the fluid velocity, $\varepsilon = 2\pi m$, and for brevity the functions $L(\xi, \zeta_1)$, $L(\xi, \zeta_2)$ have been introduced with

$$L(\xi, \zeta_i) = \tfrac{1}{2}\xi\left\{\frac{1}{\xi - \zeta_i} + \frac{1}{\xi - \bar{\zeta}_i}\right\}.$$

Equation (1.6) shows that the fluid velocity is directed along the tangent to the free surface as is required and also gives an expression for the fluid speed $q$. The parameter $\varepsilon$ is seen to be the ratio of the flux due to the source to the mainstream flux and controls the thickness of the Rankine body. From this stage it is more convenient to work with the variable $r = -1/\pi \log \xi$, which is the free-surface velocity potential due to the sink at the origin, rather than $\xi$. Equations (1.4) and (1.6) are substituted into the derivative of the Bernoulli equation (1.2) and after a little algebra the free-surface condition can be put in the form

$$F^2\{1-\varepsilon L(r,\zeta_1)+\varepsilon L(r,\zeta_2)\}\Big\{\pi\varepsilon e^{-\pi r}Q(r,\zeta_2)-\pi\varepsilon e^{-\pi r}Q(r,\zeta_1)+ + \frac{dP}{dr}\{1-\varepsilon L(r,\zeta_1)+\varepsilon L(r,\zeta_2)\}\Big\} - e^{3P}\sin\bar{\theta}=0, \quad (1.7)$$

where

$$\bar{\theta}(r)=\theta(\xi), \quad Q=\frac{dL}{d\xi} \quad \text{and} \quad P=-\int_{s=-\infty}^{\infty}\frac{\bar{\theta}(s)\,ds}{e^{\pi(s-r)}-1}.$$

This is an exact formulation of the free-surface flow around a submerged Rankine body as a nonlinear integrodifferential equation for the free-surface angle $\bar{\theta}$. Equation (1.7) can only be used in a semi-inverse manner to describe the flow around a general submerged body, as the specifications of the four quantities $F^2$, $\varepsilon$, $\zeta_1$, $\zeta_2$ which appear in (1.7) do not describe the body geometry in the physical plane but give rise to a body geometry which is only determined after the solution of (1.7) and integration of (1.3) subject to $z(\zeta_1)=id$, $z(\zeta_2)=s+id$.

Finally the determination of the body shape is carried out by first finding the position of one of the stagnation points $S'_\pm$ in the $\zeta$-plane by solving $dw/d\zeta=0$ which leads to the equation

$$-\frac{1}{\zeta}+\frac{\varepsilon}{2}\left\{\frac{1}{\zeta-\zeta_1}+\frac{1}{\zeta-\bar{\zeta}_1}-\frac{1}{\zeta-\zeta_2}-\frac{1}{\zeta-\bar{\zeta}_2}\right\}=0. \quad (1.8)$$

This quartic equation has two roots, $\zeta_s^+$, $\zeta_s^-$ say, in the upper half-$\zeta$-plane and two corresponding conjugate roots in the lower half-plane. Substitution of either of these into the expression (1.5) for the complex potential gives the value of the stream function on the body whose equation is then determined as the solution of

$$\operatorname{Im}\{w(\zeta)-w(\zeta_s^\pm)\}=0. \quad (1.9)$$

Because of its analytic complexity this equation is solved numerically to produce the coordinates of the body shape in the $\zeta$-plane. These are then transformed to the physical $z$-plane using an integrated version of (1.3):

$$z=-\frac{1}{\pi}\int_{t=-1+i0}^{\zeta}\exp\left\{-\int_{s=-\infty}^{\infty}\frac{\bar{\theta}(s)\,ds}{te^{\pi s}-1}\right\}\frac{dt}{t}. \quad (1.10)$$

Some numerical experimentation showed that unless $\arg(\zeta_1)=\arg(\zeta_2)$ the body shape is not closed, with the front and rear stagnation points lying on different streamlines.

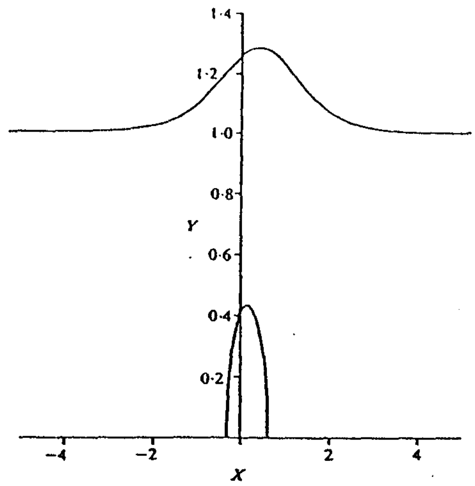


FIG. 2. Supercritical flow with $F=2{\cdot}0$, $\epsilon=2{\cdot}0$, $\zeta_1=-1+i0$, $\zeta_2=-0{\cdot}5+i0$

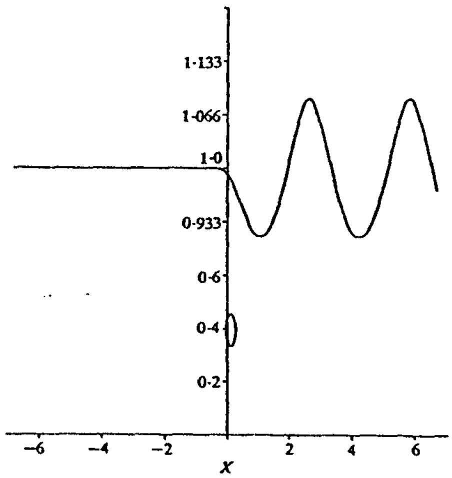


FIG. 3. Subcritical flow with $F=0{\cdot}7$, $\epsilon=0{\cdot}20$, $\zeta_1=-0{\cdot}309+i0{\cdot}951$, $\zeta_2=-0{\cdot}141+i0{\cdot}434$

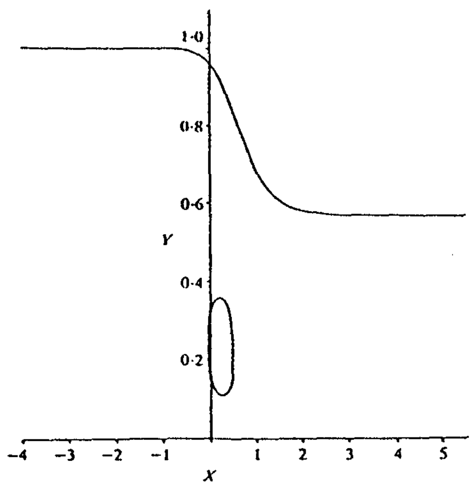


FIG. 4. Transcritical flow with $\zeta_1=-0{\cdot}707+i0{\cdot}707$, $\zeta_2=-0{\cdot}146+i0{\cdot}146$ and $\epsilon=0{\cdot}50$

References


(1) M.I.G. Bloor. J. Fluid. Mech. 84, p.167 (1978).
(2) A.C. King and M.I.G. Bloor. J. Fluid. Mech. 182, p.193 (1989).
(3) A.C. King and M.I.G. Bloor. J. Aus. Math. Soc. B 30, p.147 (1988).

## DISCUSSION

**Peregrine**: What are the conditions for the streamlines to form a closed body around the source and sink?

**King**: The condition is

$$\arg(\zeta_1) = \arg(\zeta_2). \tag{A}$$

It is easy to show that there is no flux of fluid between the source and sink if (A) holds. This implies that there is a closed streamline surrounding the Rankine body formed by the source/sink combination.

**Peregrine**: Have you calculated the force on an obstacle?

**King**: Yes. A formula for the drag on the body is found by considering the momentum flux through a control volume which starts far upstream of the body, where the flow is uniform, and ends far downstream, where the flow is either uniform (supercritical case) or one-dimensional such as under a wave crest or trough (subcritical case). In the transcritical case, the drag on the body is precisely the maximum wave drag predicted by Benjamin & Lighthill [1].

**Peregrine**: Have you calculated flows over long slender obstacles?

**King**: Yes. By letting the sink position go to infinity, I have obtained body shapes which are similar to a semi-infinite step on the bottom.

**Peregrine**: Have you any indications that finite obstacles may exist with zero waves and drag?

**King**: Yes. Linear theory predicts that when the body length is a multiple of the linear wavelength, given by $F^2 k = \tanh k$, the wave amplitude is zero. In the numerical nonlinear calculations, it was found that changing the body's length resulted in the wave drag on the body passing through a sequence of maxima and minima. The minima, although small, were never zero.

**Wu**: Please comment on your work in relation to Tuck's paper [2] on dipoles submerged beneath a free surface.

**King**: As I recall, Tuck showed that at leading order the body formed by a dipole was closed, but at second order it was not. Presumably, if Tuck's work is extended to higher orders, a different conclusion may be reached.

**Wu**: If I give you a distribution of sources and sinks, can you tell me if it will generate a closed body?

**King**: In my work, the body is closed if (A) holds. This is really a symmetry condition in the transform plane. I would expect that these symmetry results would apply to more complicated source/sink distributions.

### References


[1] T.B. Benjamin & M.J. Lighthill, 'On cnoidal waves and bores', *Proc. Roy. Soc.* A**224** (1954) 448–460.

[2] E.O. Tuck, 'The effect of non-linearity at the free surface on flow past a submerged cylinder', *J. Fluid Mech.* **22** (1965) 401-414.