Thirty years after... : the evaluation of the single integral part of the Kelvin wave source potential in the far-field

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Consider the Kelvin wave source. We define the following coordinates: the origin is taken at the position of the image source above the free surface, x is positive in the direction of forward motion, z is positive downwards; the coordinates are nondimensionalized by gravity and the forward velocity. We will use the cylindrical coordinates (x, ρ, α) with $z + iy = \rho e^{i\alpha}$, and the spherical coordinates (R, θ, φ) with $R = \sqrt{x^2 + \rho^2}$, $\theta = \tan^{-1}(\rho/x)$, $\varphi = \alpha$.

We consider the potential of the Kelvin wave source under the form used by Bessho [1], Newman [2] and Baar & Price [3]. Under this form efficient algorithms have been found for the numerical evaluation of the double integral [2] and the single integral for parts of its domain of definition. There exists several expressions for the latter, valid in more or less complementary domains:

- Bessho's convergent series ([1], eq. (4.2)) – which we note (CS1) – for a region extending from the far-field to the near-field but away from the x-axis ([3] and [5])

- Bessho's asymptotic series ([1], eq. (4.3)), completed by Ursell [6], – noted (AS1) – for the vicinity of the x-axis but away from the source ([4] and [5]), which is complementary to the region of validity of (CS1) for $1 \leq |x| \leq 16.75$ with six digits absolute accuracy

- a modified 'Bessho-Ursell' asymptotic series [5] – noted (AS3) – complementary of (CS1) and (AS1) in the neighborhood of the source.

Therefore, in order to complete this numerical evaluation, we have to find expressions for the single integral in the remaining portion of its domain of definition, namely far away from the source in a region including the cusp lines $|\theta| = \sin^{-1}(1/3) = \theta_c$.

This region was historically the first to have drawn attention from hydrodynamicists. In his 1887 paper "On Ship Waves", Lord Kelvin defined the problem satisfied by the potential and used his principle of stationary phase to show that the characteristic wave-pattern lies in the region $|\theta| \leq \theta_c$. His results, valid many wavelengths behind the source, were subsequently refined by Havelock in 1908, Hogner in 1923, Peters in 1949 and Ursell in 1960 (for a more detailed account of these results see [7]). Ursell completed the works of Hogner and Peters who used the method of steepest descents, by considering the neighborhood of the cusp line and applying there the method of Chester, Friedman & Ursell [8].

These earlier contributions were concerned with the free surface elevation, that is when both the source and the field point are on the free surface ($|\varphi| = \pi/2$). Here we propose to apply the method of steepest descents to the potential for any submergence of the source ($|\varphi| \leq \pi/2$) rather than just for the wave elevation.

As found by Hogner and Peters, the nature of the asymptotic expressions changes completely when θ passes through θ_c. This corresponds in fact, as noted by Ursell [7], to the coalescence of two saddle points, or cols, for $\theta = \theta_c$. We shall see how these results extend to the complete range of (θ, φ), and how they can be applied to a numerical evaluation.

The method of steepest descents applied to the single integral

We start from the expression for the single integral f given by:

$$f(x,\rho,\alpha) = -\frac{e^{-\frac{z}{2}}}{2}\Re\{\frac{\partial}{\partial x}F(x,\rho,\alpha)\}$$

where: $F = \frac{1}{2}(F^+ + F^-)$ and:

$$F^{\pm} = \int_{-\infty+i\alpha/2}^{\infty+i\alpha/2} \exp[-\frac{\rho}{2}\cosh(2u - i\alpha) \pm ix\cosh u]\,du$$

(see [6], eq. (2.2); [5], eq. (II.3.1), (II.3.2)).

Since the complex conjugate of F^+ is F^-, we only need to treat F^+. Using the spherical coordinates (R,θ,φ) and the change of variables $\varsigma = \exp[-u + i\frac{\varphi}{2}]$, we get:

$$F^+(R,\theta,\varphi) = \int_0^{\infty} \exp[R\cdot\chi(\varsigma,\theta,\varphi)]\,\frac{d\varsigma}{\varsigma}$$

with: $\chi(\varsigma,\theta,\varphi) = \sin\theta\cdot\big(-\varsigma^2 - 2a^*\varsigma + 2a\varsigma^{-1} - \varsigma^{-2}\big)/4$ where $a = i\cot\theta\cdot e^{i\varphi/2}$.

We want to obtain expansions for F^+ when R is large, $0 < \theta < \pi/2$, $0 \leq |\varphi| \leq \pi/2$, and for this purpose apply the method of steepest descents.

Finding the cols

The cols are given by the roots of a quartic polynomial and therefore are quite easily obtained using analytical expressions. From the location of the roots for the range of interest of (θ,φ), it appears that only two of the four roots have to be considered.

The nature of the expansions

The nature of the expansions, that is whether the expansions are in terms of Gamma functions or Airy functions, is determined by the distance between the two cols, or between their image by the mapping $\mathcal{W} : \varsigma \mapsto w = \chi(\varsigma)$. It can be shown that the coalescence of the cols only occurs for $(\theta, \varphi) = (\theta_c, \pi/2)$. From Chester, Friedman & Ursell [8] we then know that there exists a non-empty neighborhood of $(\theta_c, \pi/2)$ in $\mathbf{C}^2$ where the expansions in terms of Airy functions are uniformly valid with respect to (θ, φ). Outside this neighborhood the 'usual' expansions in terms of Gamma functions are used at each relevant col.

The relevant cols

This is the most difficult task since we are dealing with a whole set of functions $\chi(\varsigma, \theta, \varphi)$. Our goal is to define, according to certain criteria, domains of the (θ, φ)-space identified by the cols which are relevant. The method of conformal mapping developped by Ursell [9] is here very useful and allows us to define two domains: one where only one of the cols is relevant, the other where the two cols are relevant.

This defines three domains on the boundaries of which, the matching of these expansions determines the minimal accuracy that can be expected. Of course the overall numerical validity of these expansions depends on their matching with the already existing expressions (CS1) and (AS1).

The numerical implementation

The principles allowing us to obtain the two kinds of expansions are given in [8]. Due to the relative complexity of the expression of χ, the expansions are obtained up to an arbitrary order following these procedures. A series of symbolic operations, which are indeed valid for any function χ, is first performed. The resulting information is stored once and for all, then for each value of (R, θ, φ) the coefficients of the expansions are obtained according to the characteristics of the cols (if they are relevant or not) and of the expansions, from this information.

This should allow us to compute efficiently and with the desired accuracy the Kelvin wave source potential *everywhere* in the fluid domain, its integrals (with a gain in accuracy and/or speed) or its derivatives (with a loss of accuracy (speed) and/or a deterioration of the domain of validity). Exact solutions for simple cases of the thin ship theory can thus be obtained after some analytical work. By its accuracy and its efficiency this set of numerical evaluations for the potential seems to be most appropriate for wave-pattern predictions in the frame of a hybrid numerical approach for forward motion problems. But one should certainly not expect from these evaluations any kind of 'miracle' regarding a numerical solution of the Neumann-Kelvin problem for surface piercing bodies. In any attempt of this kind, it seems highly recommandable, if not absolutely necessary, to consider analytic integrations of the potential on panels or segments, rather than any kind of numerical integration of these quantities.

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REFERENCES

1. Bessho, M.,"On the Fundamental Function in the Theory of the Wave-Making Resistance of Ships", *Mem. of the Defense Academy of Japan*, **4**, **2**, 1964, pp. 99-119.

2. Newman, J. N., "Evaluation of the Wave-Resistance Green Function: Part 1-The Double Integral", *J. of Ship Research*, **31**, 2, 1987, pp. 79-90.

3. Baar, J. J. M. and Price, W. G., "Evaluation of the wavelike disturbance in the Kelvin wave source potential", *J. of Ship Research*, **32**, 1, 1988, pp. 44-53.

4. Newman, J. N., "Evaluation of the Wave-Resistance Green Function near the Singular Axis", 3^{rd} Int. Workshop on Water Waves and Floating Bodies, Woods Hole, 17-20 April 1988.

5. Clarisse, J.-M., "On the numerical evaluation of the Neumann-Kelvin Green function", M. S. thesis, MIT, January 1989.

6. Ursell, F., "On the theory of the Kelvin ship-wave source: asymptotic expansion of an integral", *Proc. R. Soc. Lond.*, A **418**, 1988, pp. 81-93.

7. Ursell, F., "On Kelvin's ship-wave pattern", *J. Fluid Mech.*, **8**, 1960, pp. 418-431.

8. Chester, C., Friedman, B. & Ursell, F., "An extension of the method of steepest descents", *Proc. Camb. Phil. Soc.*, **53**, 1957, pp. 599-611.

9. Ursell, F., "Integrals with a large parameter: paths of descent and conformal mapping", *Proc. Camb. Phil. Soc.*, **67**, 1970, pp. 371-381.

DISCUSSION

Standing: Can you clarify the source of the high-frequency oscillations in your results? Have they something to do with using the method of steepest descent, or are they an inherent feature of the source wave pattern when the source is at the surface? Many years ago, we observed something similar when computing the wavemaking of hovercraft. We were unsure whether they were due to the numerical approach we had adopted, which was similar to the method of stationary phase with numerical fade factors either side of the stationary phase points.

Clarisse: These high-frequency oscillations *are* features of the Kelvin wave source potential when the source point and the field point are both on the free surface. They correspond to the diverging waves.

Tuck: When working on the study reported in [1], one of my co-authors was rather contemptuous of my interest in the kind of steepest descents analysis you use. In the end he was able to use clever numerical equivalents of these methods to evaluate the integral directly (see Appendix 1 in [1]).

Clarisse: I agree that appropriate numerical methods can do as well as, if not better than, this approach in terms of speed, but the idea here was to obtain an evaluation which was 'uniformly' valid in the (R, θ, φ)-space (and not only for $|\varphi| = \pi/2$), consistent with analytic expressions enabling differentiations or integrations of the potential and the control of the validity and accuracy of their numerical evaluations.

Reference

[1] E.O. Tuck, J.I. Collins & W.H. Wells, 'On ship wave patterns and their spectra', *J. Ship Res.*, **15** (1971) 11–21.