# Finite-amplitude Edge Waves

T. R. Akylas & J. Mathew
Department of Mechanical Engineering
Massachusetts Institute of Technology
Cambridge, MA 02139, USA

Consider the classical problem of water waves on a uniformly sloping beach of angle $\alpha$. A coordinate system is chosen such that $x$, $y$ and $z$ are the offshore, vertical and longshore coordinates, respectively. According to linear theory (see Whitham (1979) for details), this problem admits two distinct types of wave solutions: first, a discrete spectrum consisting of a finite number of edge-wave (trapped) modes which propagate along the shore and decay seawards,

$$\Phi = F^{(n)}(x,y)\cos(kz-\omega t)$$

with $\omega^2 = k\sin(2n+1)\alpha,$

$$F^{(n)}(x,y) \sim \exp\left[k(y\sin(2n+1)\alpha - x\cos(2n+1)\alpha)\right] \quad (x\to\infty),$$

where $\Phi$ is the velocity potential; the $n$th edge-wave mode is possible if $\alpha < \pi/2(2n+1)$. Secondly, there is a continuous spectrum representing waves obliquely incident and reflected on the beach:

$$\Phi = f_l(x,y)\cos(kz-\omega t) \quad (l>0)$$

with $\omega^2 = \sqrt{k^2+l^2} = \lambda, \quad f_l(x,y) \sim \mathrm{e}^{ilx+\lambda y} + \text{c.c.} \quad (x\to\infty).$

In the present paper, we study finite-amplitude effects on edge-wave modes using perturbation expansions, similar to the Stokes expansion for periodic waves on water of uniform depth. Thus, for the $n$th mode, we write

$$\Phi = \{F^{(n)}(x,y)\mathrm{e}^{i\theta} + \text{c.c}\} + \epsilon\{S^{(n)}(x,y)\mathrm{e}^{2i\theta} + \text{c.c}\} + \cdots,$$

where $\theta = kz - \omega t$, and $\epsilon \ll 1$ is a measure of nonlinearity. Substituting the above expansion into the water-wave equations, it is found that the second harmonic $S^{(n)}$ satisfies a forced boundary-value problem of the form:

$$S^{(n)}_{xx} + S^{(n)}_{yy} - 4k^2 S^{(n)} = 0 \quad (-x\tan\alpha < y < 0),$$
$$S^{(n)}_{y} - 4\omega^2 S^{(n)} = R^{(n)}(x) \quad (y=0),$$
$$S^{(n)}_{x}\sin\alpha + S^{(n)}_{y}\cos\alpha = 0 \quad (y = -x\tan\alpha),$$

where $R^{(n)}(x)$ is a known forcing term owing to the nonlinear self-interaction of the first harmonic. A formal solution of this problem is obtained as an eigenfunction expansion in terms of the discrete and the continuous spectrum:

$$S^{(n)} = \sum_m C^{(n)}_m F^{(m)} + \int_0^\infty C_l f_l \,\mathrm{d}l,$$

where

$$C_m^{(n)} = \frac{1}{\lambda_m - 2\lambda_n} \int_0^\infty R^{(n)}(x) F^{(m)}(x,0)\ \mathrm{d}x,$$

$$C_l = \frac{1}{((2k)^2 + l^2)^{1/2} - 2\lambda_n} \int_0^\infty R^{(n)}(x) f_l(x,0)\ \mathrm{d}x$$

with $$\lambda_n = 2k \sin(2n+1)\alpha.$$

It is important to note that $C_l$ has a pole on the real $l$-axis when

$$16k^2 \sin^2(2n+1)\alpha = 4k^2 + l^2$$

for $l$ real, and for the $n$th mode this is possible in the angle range

$$\frac{\pi}{6(2n+1)} < \alpha < \frac{\pi}{2(2n+1)}.$$

Therefore, in this range of beach slopes, the second harmonic $S^{(n)}e^{2i\theta}$ is expected to have an oscillatory behaviour at infinity ($x \to \infty$), implying that nonlinear effects cause some energy to be radiated to deep water and the wave cannot remain trapped. In fact, a similar argument indicates that, for the $n$th mode, the $r$th harmonic will not be trapped if

$$\frac{\sin^{-1}(\frac{1}{r})}{2n+1} < \alpha < \frac{\pi}{2(2n+1)}. \tag{1}$$

So, the Stokes expansion suggests that all edge-wave modes will leak some energy to deep water however small the slope angle $\alpha$; of course, as $\alpha \to 0$, this effect shows up at a higher order in the expansion and, therefore, it is weaker. We note that the fundamental mode ($n = 0$, the Stokes edge wave) turns out to be exceptional; the forcing term $R(x)$ happens to vanish for both the second and third harmonic ($r = 2, 3$), as shown previously by Whitham (1976), but radiation is still expected to occur for the fourth harmonic ($r = 4$) if $\sin^{-1}(\frac{1}{4}) < \alpha < \pi/2$. We have studied the second edge mode (n $= 1$) in detail for $\alpha = \pi/8$, and have demonstrated that radiation indeed takes place at second order ($r = 2$), in accordance with (1). Also, it is noteworthy that shallow-water theory fails to predict this leaking of energy; this is understandable since shallow-water theory is not valid far from the beach, however small the beach slope $\alpha$.

The results presented above apply to edge waves on a uniformly sloping beach only. It would be of interest to know if similar restrictions apply to edge waves in the presence of more general depth variations. This problem is under current investigation.

*References*

Whitham, G. B. 1976 *J. Fluid Mech.* **74**, 353–368.

Whitham, G. B. 1979 *Lectures on Wave Propagation*, Tata Institute of Fundamental Research, Bombay, Springer.

## DISCUSSION

**Palm**: The free waves at large values of $x$ are harmonic functions of $x$ and $z$. It is possible for the energy to be propagated in various directions. Which direction is preferred? If the wave direction is oblique, what is the energy source for the energy transport *along* the shore?

**Akylas**: Energy propagates along the shore owing to the edge waves and also away from the shore owing to the radiated oblique wave. The direction of radiation can be found explicitly, and, for a uniformly sloping beach, happens to be independent of the edge wavenumber. After a long time, the edge wave amplitude will decay and therefore the energy transport diminishes as well.

**Peregrine**: Have you considered standing edge waves?

**Akylas**: The radiation damping of *standing* edge waves has been investigated in earlier work by Guza & Bowen [1], Minzoni & Whitham [2], and others. In this case, radiation damping is possible for all beach slopes and can be studied using the shallow-water equations. For progressive edge waves, however, radiation damping is possible only for certain ranges of beach slopes, and *cannot* be studied using shallow-water theory.

### References


[1] R.T. Guza & A.J. Bowen, 'Finite amplitude edge waves', *J. Mar. Res.* **34** (1976) 269–293.

[2] A.A. Minzoni & G.B. Whitham, 'On the excitation of edge waves on beaches', *J. Fluid Mech.* **79** (1977) 273–287.