REGULAR AND CHAOTIC WAVE MOTIONS IN A RECTANGULAR WAVE TANK

Wu-ting Tsai and Dick K.P. Yue

Department of Ocean Engineering

MIT, Cambridge, Mass., USA

We examine the resonantly excited longitudinal and transverse (respectively perpendicular and parallel to the wavemaker) motions in a three-dimensional rectangular wave tank with a harmonically driven wavemaker on one side. Depending on the length and width of the tank relative to the forcing frequency and water depth, *i.e.*, on the degree of longitudinal (synchronous) and transverse (subharmonic) turning, the possible orders of magnitude of the longitudinal and transverse wave motions relative to wave paddle amplitude are systematically considered. Specifically, the following three sets of ordering can be identified:

	wavemaker	longitudinal wave	transverse wave
case I	$O(\varepsilon)$	$O(\varepsilon^{1/3})$	$O(\varepsilon^{2/3})$
case II	$O(\varepsilon)$	$O(\varepsilon)$	$O(\varepsilon^{1/2})$
case III	$O(\varepsilon)$	$O(\varepsilon^{1/2})$	$O(\varepsilon^{1/2})$

where ε is the nondimensional amplitude of wavemaker oscillation normalized by the length of the tank, and $\varepsilon \ll 1$.

Cases I and II correspond respectively to the well-known synchronous longitudinal forced-excited and subharmonic transverse parametric-excited standing waves studied by a number of investigators (*e.g.*, Lin & Howard, 1960; Garrett, 1979; Miles, 1988) notably in deep water. Case III is new, and involves large resonant longitudinal and cross waves of the same order of magnitude. In this case, internal interaction or resonance between the two is also an important mechanism.

Synchronous Forced-Resonant Longitudinal Waves – Case I

If the excitation frequency of the wavemaker is approximately equal to a natural frequency of longitudinal standing wave in the tank (say, the n-th harmonic mode), but the length-to-width ratio ℓ is not close an integral multiple of 1/4, then only the longitudinal wave is resonantly excited by the wavemaker. The appropriate asymptotic approximation for this case for the velocity potential Φ and free surface elevation ς are perturbation series in powers of $\varepsilon^{1/3}$. Choosing the long time scale $\tau = \varepsilon^{2/3}t$, and introducing a tuning factor for the excitation frequency, $\Omega_x/\omega_e = 1 + \varepsilon^{2/3}\lambda_x$, where Ω_x is the dimensional *linearized* natural frequency of the longitudinal standing wave mode, ω_e is the dimensional frequency of the wavemaker oscillation, and λ_x is the detuning parameter. Carrying out the perturbation systematically, we obtain at third order ($O(\varepsilon)$) (in order to suppress the secularity), an evolution equation for the complex amplitude $A(\tau)$ of the velocity potential of the longitudinal motion:

$$\mu_x \frac{dA}{d\tau} + i2\lambda_x\mu_x A + \frac{\delta}{4} - i\Gamma_a A^2 A^* = 0, \tag{1}$$

where μ_x, Γ_a and δ are functions of depth h and mode number n. The amplitude of the longitudinal stationary wave and the stability of this response ares obtained readily from the evolution equation (1) and compare favorably with the experiments of Lin & Howard (1960).

At a critical depth h^*, Γ_a in (1) changes sign from positive to negative as the depth h decreases and the longitudinal standing wave switches from a softening to hardening spring system. At $h = h^*$, $\Gamma_a = 0$ and the perturbation analysis above breaks down. Thus in the neighborhood of h^*, we choose the magnitude of longitudinal waves as $\varepsilon^{1/5}$, and define the tuning factor as $\Omega_x/\omega_e = 1 + \varepsilon^{4/5}\lambda_x$. Carrying out the perturbation analysis this time through five orders, we obtain a new evolution equation which contains a quintic nonlinearity and is valid for $| h - h^* | \leq 0(\varepsilon^{2/5})$ for $\lambda_x \leq 0(1)$.

Subharmonic Parametric-Resonant Transverse (Cross) Waves – Case II

This is the case when the excitation frequency is close to a superharmonic frequency of a cross-tank resonant mode while the longitudinal waves are not resonantly excited. Following the experiments and analysis of Lin & Howard (1960) for the problem, Garrett (1970) confirmed that the mechanism for cross-wave excitation is indeed parametric resonance by obtaining the linear Mathieu's equation governing the amplitude of the transverse waves. Recently, Miles (1988) used a Lagrangian formulation and obtained a Hamiltonian system governing the slow modulation of the amplitude of transverse waves. Accordingly, for 1/2 subharmonic transverse waves with order of magnitude $\varepsilon^{1/2}$, we choose the long time scale $\tau = \varepsilon t$ and define the cross-wave detuning parameter λ_y as $\Omega_y/\omega_e = 1/2 + \varepsilon\lambda_y$. The length-to-width ratio ℓ is assumed to be far from an integral multiple of 1/4 such that the longitudinal wave is not resonantly excited.

Carrying the perturbation procedure, and applying the solvability condition at third order, we obtain an evolution equation for the complex amplitude $B(\tau)$ of the transverse waves:

$$\mu_y \frac{dB}{d\tau} + i2\lambda_y\mu_y B - i\beta B^* - i\Gamma_b B^2 B^* = 0, \tag{2}$$

where μ_y, Γ_b and β are functions of h and ℓ. The coefficient β of B^* is the result of parametric resonance and contains terms resulting from both the first-order (equivalent to Garrett's linear result obtained by averaging the longitudinal wave motion) and second-order (due to direct interaction between the wavemaker motion and the transverse waves) wavemaker boundary conditions. Equation (2) is isomorphic to equation (4.1) of Miles (1988) after a $\pi/4$ phase shift of his complex amplitude. The stationary solutions (and their local statbility) are computed and are again in good agreement with Lin & Howard's measurements. Again, we note that there exists $h = h^{**}$ where $\Gamma_b(h^{**}) = 0$, and the weakly nonlinear analysis described above breaks down. To obtain a uniformly valid description, we expand Φ and ς in powers of $\varepsilon^{1/4}$ for h near h^{**}, and carry out the perturbation analysis to $O(\varepsilon^{5/4})$ resulting in a new equation with a quintic interaction term.

Interaction between Resonant Longitudinal and Transverse Waves – Case III

When the excitation frequency of wavemaker is approximately equal to the natural frequency of the longitudinal n-th harmonic standing wave and the length-to-width ratio ℓ is close to n/4 (for first-mode cross-waves), the longitudinal wave is directly resonanted by the wavemaker while the transverse wave is parametrically resonanted. Both waves are the same order of magnitude $O(\varepsilon^{1/2})$, and interact with each other. To account for the two resonances which are involved at different orders, two long time scales are introduced: $\tau_1 = \varepsilon^{1/2}t$ and $\tau_2 = \varepsilon t$. The degree of tuning of the wavemaker with respect the longitudinal and cross waves are characterized by $\Omega_y/\omega_e = 1/2 + \varepsilon^{1/2}\lambda$, and $\Omega_x/\Omega_y = 2 + \varepsilon^{1/2}\gamma$, where λ and γ are the detuning parameters.

Carrying out the perturbation systematically through orders $0(\varepsilon^{1/2})$, $0(\varepsilon)$, and $0(\varepsilon^{3/2})$, and combining the solvability conditions at the second and third orders, we obtain finally two coupled evolution equations governing the amplitudes of the longitudinal and transverse motions:

$$\mu \frac{d\tilde{A}}{d\tau} + i\gamma_a \tilde{A} + \tilde{\delta} - i\tilde{\Gamma}_a \tilde{A}^2 \tilde{A}^* - i\tilde{\Sigma}\tilde{A}BB^* = 0, \tag{3}$$

and

$$\mu\frac{dB}{d\tau} + i\gamma_b B - i\tilde{\beta}B^* - i\tilde{\Gamma}_b B^2 B^* - i\tilde{\Sigma}B\tilde{A}\tilde{A}^* = 0, \tag{4}$$

where the modulation of the forcing has been factored explicitly in A by letting $A \equiv \sqrt{2}\tilde{A}e^{i\gamma\epsilon\tau}$, $\tilde{A}$ and B are now considered functions of τ_1 only, *i.e.*, $(\partial/\partial\tau_1) + \varepsilon^{1/2}(\partial/\partial\tau_2) \rightarrow (\partial/\partial\tau)$; $\mu = \mu_y = 4\mu_x + O(\varepsilon^{1/2})$; $\gamma_{a,b}$ are functions of h, λ, γ, and ε; $\tilde{\delta}$ is a function of n, h, λ, γ and ε; $\tilde{\beta}$ is a function of ℓ, h, and ε; $\tilde{\Gamma}_a = 8\varepsilon^{1/2}\Gamma_a$; $\tilde{\Gamma}_b = \varepsilon^{1/2}\Gamma_b$; and $\tilde{\Sigma}$ is a function of h, ℓ, n and ε. Equations (3,4) reduce to equations (1,2) in the absence of resonant (transverse,longitudinal) wave motions.

If we write $\tilde{A} \equiv C_a + iD_a$ and $B \equiv C_b + iD_b$, then equations (3), (4) can be represented as an autonomous Hamiltonian system with the Hamiltonian $\mathcal{H}$ given by:

$$\begin{aligned}\mathcal{H} =& \frac{1}{\mu}\Big[\tilde{\delta}D_a - \frac{1}{2}\gamma_a(C_a^2 + D_a^2) + \frac{1}{4}\tilde{\Gamma}_a(C_a^2 + D_a^2)^2 \\ &+ \frac{1}{2}\tilde{\beta}(C_b^2 - D_b^2) - \frac{1}{2}\gamma_b(C_b^2 + D_b^2) + \frac{1}{4}\tilde{\Gamma}_b(C_b^2 + D_b^2)^2 + \frac{1}{2}\tilde{\Sigma}(C_a^2 + D_a^2)(C_b^2 + D_b^2)\Big],\end{aligned} \tag{5}$$

where C_a, D_a and C_b, D_b are conjugate variables which satisfy the Hamilton's equations

$$\frac{dC_{a,b}}{d\tau} = -\frac{\partial\mathcal{H}}{\partial D_{a,b}}, \qquad \frac{dD_{a,b}}{d\tau} = \frac{\partial\mathcal{H}}{\partial C_{a,b}}. \tag{6}$$

Stationary solutions and their local stability are readily calculated for the evolution equations (6). The bifurcation of the stationary solution amplitude varying with the excitation detuning parameter λ changes abruptly around the intermediate depths, $h = 1.5 \sim 1.9$. While for h greater than 2.5, the bifurcation diagrams are qualitatively all similar. The three-dimensional wave family starts at the pitchfork bifurcation point on the two-dimensional longitudinal wave branch, where the two-dimensional wave family loses stability and a pair of pure imaginary eigenvalues separate into two pairs along the three-dimensional branch. For the cases of intermediate depths, the so-called Hamiltonian-Hopf bifurcation occurs on the three-dimensional wave family. Stability of the three-dimensional wave family is lost at such a bifurcation point where the two pairs of pure imaginary eigenvalues collide in pairs again and then split into two complex conjugate pairs leaving the imaginary axis. Such bifurcations correspond to Benjamin-Feir instabilities in two-dimensional steady progressive wave.

Numerical integrations are performed for the interaction equations (6). The results of temporal simulations exhibit either regular (periodic and quasi-periodic) or chaotic behaviors, depending upon the parameters and initial conditions. For the chaotic evolutions, two solutions with slightly different initial conditions separate at an exponential rate, and small differences in initial conditions are manifested at a later time by vastly different dynamical states. Such characteristic sensitivity to initial conditions can be precisely quantified in terms of the Lyapunov characteristic exponent which measures the mean rate of exponential separation of neighboring evolution trajectories. The scheme suggested by Benettin *et al.* (1976) is applied to compute the largest Lyapunov exponent indicating that the trajectories with finite exponents are indeed chaotic. Another characterization for regular and chaotic behaviors is the power spectrum of the evolution amplitude. Power spectra of the amplitudes obtained by fast Fourier transforms of simulated evolutions again confirm the occurrence of chaotic motions.

Resonance Overlapping as a Criterion for the Onset of Widespread Chaos

To understand the global behavior of the Hamiltonian system (6) in phase space, we construct the two-dimensional first return map on the hypersurface $\Sigma_{\mathcal{H}}$ of codimension one. Such a hypersurface is known as a Poincaré surface of section. In our problem the surface of section is chosen as

$$\Sigma_{\mathcal{H}} = \{(C_a, D_a, C_b, D_b): \ C_a = 0, \ \frac{dC_a}{d\tau} > 0, \ \mathcal{H} = \mathcal{H}(C_a, C_b, D_a, D_b; \lambda, h, \ell, n)\}. \tag{7}$$

On the surface of section, a fixed point corresponds to a periodic trajectory, successive points lying on some smooth curves (invariant curves) belong to a quasi-periodic orbit, while those belonging to chaotic orbit appear to fill a regime.

According to KAM theorem (cf. Chirikov, 1979), for an integrable system, those invariant curves with sufficiently incommensurate winding numbers persist under small perturbations. As the strength of perturbation increases, neighboring resonance zones will interact and chaotic motion is confined to a narrow regime around the separatrices bounding the resonance zones. As two resonance zones grow and eventually overlap, invariant curves between them will be destroyed, resulting in the onset of widespread chaos. The method of overlapping resonance developed by Chirikov (1979) postulates that the last invariant curve between two lowest-order resonances is destroyed when the sum of the half widths equals the distance between the resonance centers. A major approximation is that the width of each resonance zone can be calculated independently of all the others. This simple criterion yields an estimate for the critical parameters governing the appearance of widespread chaos.

Applying the canonical transformation: $\tilde{A} = i\sqrt{2I_a}exp(i\theta_a)$ and $B = i\sqrt{2I_b}exp(i\theta_b)$, where $I_{a,b}$ and $\theta_{a,b}$ are action and angle variables, the Hamiltonian takes the new form:

$$\begin{aligned}
\mathcal{H} &= \mathcal{H}_0 + \mathcal{H}_a + \mathcal{H}_b, \\
\mathcal{H}_0 &= \frac{1}{\mu}(-\gamma_a I_a - \gamma_b I_b + \tilde{\Gamma}_a {I_a}^2 + \tilde{\Gamma}_b {I_b}^2 + 2\tilde{\Sigma} I_a I_b), \\
\mathcal{H}_a &= \frac{\tilde{\delta}}{\mu}\sqrt{2I_a}\cos\theta_a, \qquad \mathcal{H}_b = -\frac{\tilde{\beta}}{\mu} I_b \cos\theta_b,
\end{aligned}$$

which consists of an integrable part $\mathcal{H}_0$ and two nonintegrable perturbations $\mathcal{H}_a$ and $\mathcal{H}_b$ responsible for the two primary resonances. Introducing two new resonant angle variables which change slowly relative to other variables in the Hamiltonians, and averaging the Hamiltonians over the fast variables we obtain the approximate resonant Hamiltonians. These two resonant Hamiltonians have exactly the form of the pendulum Hamiltonian whose widths of separatrices can be calculated. From these resonance widths we obtain the boundaries of resonance in the original action variables. In our computations we find that the surface of section exhibits large widespread chaos regimes if the resonance overlapping area is large.

References

Benettin, G., Galgani, L. & Strelcyn, J.M. 1976. Kolmogorov entropy and numerical experiments. *Phys. Rev.* **A14**, 2338.

Chirikov, B. 1979. A universal instability of many dimensional oscillation systems. *Phys. Rep.* **52**, 263.

Garrett, C.J.R. 1979. On cross-waves. *J. Fluid Mech.* **41**, 837.

Lin, J.N. & Howard, L.N. 1960. Non-linear standing waves in a rectangular tank due to forced oscillation. *MIT Hydrodynamics Laboratory, Rep.* **44**.

Miles, J.W. 1988. Parametrically excited, standing cross-waves. *J. Fluid Mech.* **186**, 119.

DISCUSSION

Evans: I should like to compliment you on your most impressive paper. I believe this is the first time 'chaos' has been discussed at these workshops fully 25 years after Lorenz's paper. There are of course many areas of interest where it may arise, such as the present wavewater problem, ship stability theory and models of TLPs. The Faraday problem of vertically or horizontally forced rectangular tank is clearly related to the wavewater problem. Have you followed the many papers published recently in Phys.Lett. on this problem?

Tsai & Yue: We are certainly aware of and have benefitted much from the many contributions of other workers on this class of problems. Most of these are from the applied physics and aerospace community on the horizontal or vertical vibrations of tanks. The present problem of a tank with a wavemaker is perhaps more akin to our field of water waves and floating bodies. It is gratifying that for this relatively simple problem, the solutions are every bit as sublime and rich if not more so as those in other contexts.

Palm: You show that in your Hamiltonian system you obtain periodic motion, quasiperiodic motion and chaos by only verifying the initial conditions. Don't you believe that this result would be radically changed by taking into account a (small) viscosity?

Tsai & Yue: The results will certainly be very different in the presence of dissipation. However, it seems clear that orbits which are now chaotic will still be chaotic although the converse will not necessarily be true.

Miloh: Being aware of the large effort and brain power put into this paper I want to compliment you for an excellent presentation. Together with my colleagues Drs Kit and Shemer from the Univ. of Tel-Aviv we did some theoretical and experimental studies of nonlinear sloshing and cross waves in a relatively long channel. We found that viscous wave damping on the wave maker is very important and should be included in the formulation on the form of an additional complex parameter in the non-linear Schrödinger equation. Since your tank is relatively short I believe that in your case viscous damping will play a more important role. As a result of the dissipation the system will not be Hamiltonian any more. Can you please make a comment on the possibility of incorporating damping in your calculations?

Tsai & Yue: We are aware of your work and the likely importance of dissipation in these problems. If dissipation is to be modelled, a small linear damping term, say, can be readily added to our evolution equations. Much of our computational analyses can be carried out accordingly although the system is now more complicated. The possible application of a resonance overlapping idea to a weakly dissipative problem has intrigued us for some time now, although it is still too early to give an answer to this question.