The Effect of Viscosity and Surfactant on Nonlinear Water Waves

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Although water waves are usually well described by inviscid-flow theory, the effects of viscosity and interfacial properties are important for waves of short wavelength. This, in turn, requires the neglected shear-stress boundary condition to be satisfied on the free surface. When the free surface is clean and the air above is neglected, the shear stress vanishes. However, when the water surface is covered with contaminant (or surfactant), its concentration varies in accordance with the wave motion on the free surface, causing a surface-tension gradient which must be balanced by a non-zero surface shear stress. The highly dissipative effect of surfactants has been known since classical times, as summarized by Garrett (1986). We are motivated by recent radar images of ship waves which show distinctive dark regions that might be related to the calming effect of surfactants.

In most cases, the study of viscous damping associated with water waves has been confined to linear theory. Here, we develop an analytical method for weakly nonlinear waves, which yields an evolution equation for the slowly varying amplitude. For inviscid water, the result is usually the nonlinear Schödinger equation (NLS), whereas dissipation produces a Ginzburg-Landau equation (GL) here. The GL equation enables the study of damping and instability of infinite wavetrains (the viscous equivalent of Stokes waves) and the evolution and attenuation of envelope solitons. Solutions to the GL equation can exhibit chaotic behavior under certain circumstances, which would lead to the production of short waves. However, here we emphasize the derivation of the GL equation and subsequent damping coefficients.

Since the fluid depth is much larger than the short wavelengths studied here, we consider a two-dimensional semi-infinite fluid bounded by a free surface. The fluid below the free surface has constant density ρ and viscosity μ, and the air above can be neglected. An insoluble surfactant on the free surface is characterized by the surface dilational modulus M, which is a measure of the resistance against the compression-expansion type of surface deformation (Lucassen, 1982). The flow is initiated by surface waves with small amplitude, and for small viscosity the flow is irrotational everywhere except in a thin boundary layer beneath the free surface. This boundary layer is needed to satisfy the shear-stress boundary condition on the free surface. In general, the thickness of the boundary layer is much smaller than the wave amplitude a_m. Therefore, a coordinate system attached to the free surface is desirable.

The equations that govern the flow are the momentum and the continuity equations in the boundary layer and the Laplace equation for the velocity potential in the outer irrotational region. Normal and shear stress boundary conditions on the free surface involve the pressure and the surface-tension gradient, respectively. The kinematic condition is also prescribed. When the reciprocals of the wavenumber k and the primary frequency ω_o of the wave are used as the length and time scales, the following dimensionless groups are formed:

$$\text{Reciprocal Reynolds number}: \epsilon = \frac{k^2\mu}{\rho\omega_o} \quad \text{Reciprocal Froude number}: \bar{g} = \frac{kg}{\omega_o^2}$$

$$\text{Nondimensional amplitude}: a = ka_m \quad \text{Weber number}: T = \frac{k^3\sigma}{\rho\omega_o^2}$$

$$\text{Nondimensional surface dilational modulus}: \kappa = \frac{kM}{\mu\omega_o}\epsilon^{1/2},$$

where g is the gravitational acceleration. The relative importance of nonlinearity, surfactant, and viscosity is determined through the scaling of a, κ, and the boundary-layer thickness, which is $O(\epsilon^{1/2})$. Here, we choose κ to be $O(1)$ to observe the impact of surfactants at first order. The relationship between a and the boundary-layer thickness is chosen such that the amplitude equation can be determined at third order.

In addition to a stretched variable for the method of matched asymptotic expansions, we introduce the multiple scales

$$\tau = a^2t, \quad \xi = a(x - c_g t), \quad \bar{y} = ay,$$

where c_g denotes the group velocity of the primary progressive wave and t, x, and y are the nondimensional temporal and spatial variables. Slow modulation in the vertical direction is also considered to remove the limitation on the water depth, as explained by Mei (1983) for inviscid flow.

Now it is straightforward to develop asymptotic solutions for the inner boundary layer and outer potential region expanded in terms of small amplitude a. Matching conditions are applied at each order to uniquely determine the solutions. At first order, the matching condition for the hydrodynamic pressure gives the linear dispersion relationship

$$\bar{g} + T = 1.$$

At second order, matching determines the group velocity

$$c_g = \frac{1 + 2T}{2},$$

and the third-order matching finally yields the desired amplitude equation.

For viscous gravity waves with amplitude of the same order as the boundary-layer thickness, the result is a single GL equation in its simplest form:

$$iA_\tau + 2iA - \frac{1}{8}A_{\xi\xi} = 2\,|A|^2 A,$$

where A is the slowly varying amplitude such that the free-surface elevation is described as

$$\eta = A(\xi, \tau)e^{i(x-t)}.$$

Solving the GL equation after dropping the dispersion term gives the damping coefficient 2ϵ, which is identical to the linear result of Stokes. The nonlinear damping effect can also be shown not to appear at this order from a simple analysis of the viscous dissipation terms.

When capillary effects are included at larger wave amplitude, $O(\epsilon^{1/4})$, the amplitude equation becomes

$$iA_\tau + \frac{i}{2\sqrt{2}}\,\frac{\kappa^2 + i(\sqrt{2}\,\kappa - \kappa^2)}{\kappa^2 - \sqrt{2}\,\kappa + 1}A - \frac{1}{8}(4T^2 - 8T + 1)A_{\xi\xi} = \frac{1}{4}\,\frac{9T^2 - 15T + 8}{1 - 3T}\,|A|^2 A.$$

Since $\tau = \epsilon^{1/2}t$ here, it is clear that viscous damping has been enhanced greatly by the surfactant. Again, the damping coefficient is unaltered from the linear result of Levich (1962) and others. This result confirms the qualitative experimental observation of Lucassen (1982).

Preliminary results indicate that the stability of the infinite wavetrain to a sideband disturbance of Benjamin-Feir type is modified by dissipation. We also will show some preliminary spectral computations examining the evolution of envelope solitons in the presence of surfactants. Various coordinate systems for forming the amplitude equations will be discussed.

This work was supported under the Program in Ship Hydrodynamics at The University of Michigan, funded by the University Research Initiative of ONR Contract No. N000184-86-K-0684.

REFERENCES

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DISCUSSION

Miloh: Why do you call this the Ginzburg-Landau equation and not the non-linear damped Schrödinger equation?

I wonder if the nonuniformities found in the solution for T=1 and T=1/3 have to do with the Wilton ripples resonance? In the KdV formulation we know that leading order dispersion term which appears in the form of the third derivative of the amplitude will vanish and a next order term, i.e. fifth derivative, has to be considered. Is this related to your singularity?

Joo, Messiter & Schultz: The non linear damped Schrödinger eq. may indeed be more appropriate to our amplitude eg. However, an evolution with cubic-nonlinear, dispersion and dissipation terms has been called Ginzburg-Landau eq. in the literature including Moon et al. (1983), Keete (1985) and Deisslei (1985).

The case for T=1/3 is related to Wilton's ripples. The non-uniformity can be removed by considering a superposition of a fundamental wave and its second harmonic both at first order and by adding an intermediate slow-time scale, as is done by McGoldrick (1970) for an inviscid flow. Here, an extension of his analysis enables us to get uniform results near T=1/3.