ON THE WAVE-RESISTANCE GREEN FUNCTION

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The potential of a submerged source moving with constant horizontal velocity beneath a free surface can be expressed by several well known integral representations. However, the numerical evaluation of these expressions is a major obstacle to the development of computational models for analysing ship waves and wave resistance. The source potential, or Green function, can be expressed as the sum of an elementary Rankine source 1/R, its image 1/R above the free surface, and a double integral which accounts for the linearized free-surface effects. In the double integral, which can be interpreted as a distribution in wavenumber space, a pole is encountered at the wavenumber of the free waves propagating in a steady state relative to the moving disturbance. The contour of integration may be deformed in an appropriate manner around this singularity, to proscribe waves upstream of the disturbance. Equivalently, the double integral can be expressed in terms of its Cauchy principal value, and augmented by a single integral associated with the residue at the pole ([1], eq. 13.36):

$$G = \frac{1}{R_0} - \frac{1}{R} - \frac{4}{\pi}\int_0^{\pi/2} d\theta \int_0^{\infty} dk \; \frac{e^{-kz}\cos(kx\cos\theta)\cos(ky\sin\theta)}{k\cos^2\theta - 1}$$
$$-4\int_0^{\pi/2} d\theta \;\; e^{-z\sec^2\theta}\sin(x\sec\theta)\cos(y\sec^2\theta\sin\theta)\sec^2\theta \tag{1}$$

Here the Cartesion coordinates (x,y,z) are nondimensionalized by the wavenumber corresponding to the forward velocity, which coincides in direction with the x-axis. The origin is in the free surface and the vertical z-axis is positive downwards.

Other equivalent expresions may be derived by contour integration and residue calculus, with different decompositions between the double integral and residual single integral. In one of the most effective decompositions for numerical analysis the radiating waves are described completely by the single integral, and the remaining double integral represents a symmetric local disturbance:

$$G = \frac{1}{R_0} - \frac{1}{R} - \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} d\theta \int_0^{\infty} dk \; \frac{e^{-kz + ik|x|\cos\theta + ikysin\theta}}{k\cos^2\theta - 1 + i\epsilon\cos^2\theta}$$

$$+4iH(-x) \int_{-\pi/2}^{\pi/2} d\theta \; \sec^2\theta \; e^u \tag{2}$$

where

$$u = -z\sec^2\theta + ix\sec\theta + i|y|\sec^2\theta\sin\theta \tag{3}$$

In this work a new computational approach is adopted to evaluate the double integral, in the latter decomposition, based on the use of multidimensional polynomial approximations. This technique has been described briefly [2] in the special case y=0, where the source and field point are in the same transverse plane. The essential step is to regard the double integral (minus a singular component at the origin) as a regular function of three variables which can be expanded and approximated systematically by Chebyshev polynomials.

This approach is more common for the evaluation of transcendental functions of a single variable, and useful analogies can be drawn, for example, from the usual algorithms for evaluating the Bessel function Y_0. It is necessary first to ensure that the function to be approximated is sufficiently regular, and for this purpose its analytic behavior must be understood both close to the origin, when the source and field point are coincident on the free surface, and far away. The double integral has a weak singularity at the origin, which must be removed systematically to ensure good convergence of the subsequent Chebyshev expansions. On the other hand, the asymptotic form far from the origin is useful computationally, and provides guidance in choosing the appropriate form for the approximation in that domain. The different analytic structures in these two limits dictate separate approximations, and ultimately we find it necessary to establish three partitions at constant values of the radius R, with separate polynomial approximations in each of the four resulting subdomains.

The application of the present results is extremely simple and effective. In particular, the double integral can be evaluated to five or six decimals absolute accuracy throughout the computational domain by evaluating a polynomial with about 200 coefficients. A smaller number of terms can be used if less accuracy is desired.

The singularity at the origin is studied by expressing the inner member of the double integral in terms of the complex exponential integral [2, eq. 36], and retaining only the logarithmic singularity of that function. The resulting singular component S is then derived by expanding the remaining exponential factor of the integrand in powers of the three Cartesian coordinates, and integrating term-by-term. This procedure is systematized by defining two families of definite integrals and deriving algorithms for their recursive evaluation. For the numerical results derived here, this process is truncated after including powers of the spherical radius R up to three. The resulting expression for S includes twenty terms, and requires the recursive evaluation of 12 definite integrals.

The asymptotic expansion for large values of R is in descending negative powers of R times functions of the two spherical angles. The basic form of this expansion can be deduced rigorously from the above representation involving the complex exponential integral, but the details of this expansion are more easily derived from an indirect procedure. We start with an alternative form of the Green function, which is more directly related to (1) after accounting for the branch cut of the exponential integral E_1 along the negative real axis:

$$G = \frac{1}{R_0} - \frac{1}{R} - \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \sec^2\theta \; \epsilon^u \; E_1(u)\; d\theta + 4i\int_{-\pi/2}^{-\tan^{-1}(x/|y|)} \sec^2\theta \; e^u \; d\theta \tag{4}$$

In this form the complex exponential integral has an argument which is bounded from below and proportional to R. Asymptotic expansion of E_1 is then justified, and the resulting approximation of the integrand may be integrated term-by-term. This yields an asymptotic expansion of the first integral which is regular up to the free surface, as is the original double integral in (2). Since the second integral in is exponentially small when $z>0$, the derived expansion of the first integral is asymptotically equivalent to the double integral in (2) for points up to and including the plane $z=0$. The final result of this procedure is a two-parameter family of definite integrals related to the associated Legendre functions, and thus amenable to recursive evaluation.

The derivation of triple Chebyshev expansions in the three spherical coordinates is performed after establishing radial partitions at R=1, 4, and 10. In the first subdomain the singular component S is subtracted from the double integral, and the difference is evaluated in double precision by Romberg quadratures. The discrete orthoganality relations are then used to evaluate the coefficients of the three-dimensional expansion, including Chebyshev polynomials up to the order n=16. The coefficients with absolute value exceeding 1E-9 have been retained, and the subset with values greater than 1E-6 yields an effective single-precision approximation for the double integral. A similar procedure is followed in the other subdomains, without subtraction of S. (In the domain R>10 the radial argument of the Chebyshev polynomials is a linear function of 1/R, suggested by the asymptotic expansion of the double integral.) In each of the four subdomains about 200 coefficients are retained with absolute values greater than 1E-06.

Equivalent "economized" ordinary polynomials in powers of the three spherical coordinates are derived from the truncated Chebyshev expansions. The use of these polynomials permits the double integral to be evaluated by nested multiplication, throughout the physical domain, with about 200 floating point multiplications and the same number of additions. (For the domain 0<R<1 the singular component S must also be evaluated, but this is a relatively small computational burden.)

A paper describing this analysis with tables of the Chebyshev and ordinary polynomial coefficients will be submitted for publication in the Journal of Ship Research. This work was supported by the Office of Naval Research and the National Science Foundation.

REFERENCES

1. J. V. Wehausen and E. V. Laitone, Surface waves, Handbuch der Physik 9, 446-778. Springer-Verlag, Berlin (1960).

2. J. N. Newman, The evaluation of free-surface Green functions, 4th Numerical Ship Hydrodynamics Conference, Washington, (1985).

Discussion

Tuck: When you were working out the integrals, were you using double integrals or were you using an efficient algorithm for the exponential integral?

Newman: I evaluated E_1, using methods such as the continued fraction, which I presented at the Fourth International Conference on Numerical Ship Hydrodynamics, 1985.

Papanikolaou: Referring to the use of entirely analytical expressions for Green's formula, I would like to state that in the corresponding case of the 3D Green's function or pulsating source at zero forward speed and finite water depth, Gauss-Laguerre quadratures can be very efficient in the region where John's formula cannot be used. In doing this, it is assumed that the required Bessel and exponential-integral functions are given through Chebyshev polynomial expressions.
[1] Papanikolaou, A. "On Alternative Methods for the Evaluation of Green's Function of a Pulsating 3D Source for Arbitrary Water Depth and Frequency of Oscillation", Berlin TUB/ISM rep 83/17 (1983) Submitted for publication, Journal of Eng'g Mathematics.

Newman: We have to be careful not to compare apples and oranges. You have to be sure the computation is done to equal accuracy and can be done over the entire domain.

Hearn: We are also looking for efficient algorithms for both the translating and the translating and pulsating source Green's functions.

Tuck: Now that the times to compute elements of the influence matrix can be reduced to very low values, reducing the time for the matrix inversion should be given more attention than it is currently given.

Hearn: I agree. Now we should attack the solution scheme.

Newman: I have heard some say the finite-depth problem at zero speed can be done starting with John's formulation and letting the depth go to infinity. However, it is never efficient to use a general algorithm repeatedly for a special case.

C. Lee: I think Newman's contribution is significant. Can you also do it for a doublet? Now that the first step toward improving the numerical efficiency for free-surface problems has been taken, I think we should also direct our efforts to resolve the following outstanding problems: (1) For a given free-surface problem, what type of singularity (source, doublet, or combination of both) should we use? (2) For a given geometry, type of singularity, frequency of oscillation and ship speed, how should we choose the type (quadrilateral, triangular, etc.), size and number of panels? and (3) How should we distribute the singularities for given panels; constant, linear, parabolic, or higher order polynomials? I would like to emphasize that the above problems should not be treated by a numerical trial-and-error basis, if avoidable, but rather by some clever mathematical analysis.

Newman: One feature of these polynomial representations is that they can be differentiated term-by-term with simple algorithms. Thus the dipole, and derivatives of both singularities, can be evaluated from the same set of polynomial coefficients which are presented here. The only concern to keep in mind is that some accuracy will be lost, typically on the order of one decimal per differentiation. This is one reason why six decimals are retained in the source potential.

Yue: I would like to offer a corollary to Prof. Tuck's "axiom" that the computational effort spent on the influence coefficient evaluation and the matrix inversion should be comparable. Assuming a direct solution of the equation system, we should aim to spend $0(N)$ operations on the former - with the kind of work described here, influence coefficients can be computed in much less than $0(N)$ flops (where, say, $N = 0(10^3 - 10^4)$). So indeed, we should put more of our effort into the problem of solving the equation system.

Newman: I certainly agree. In the past we have been concerned primarily about the time required to evaluate the influence coefficients, but with fast algorithms now available, and the practical need to increase the number of panels, the matrix solver will be the limiting factor. Iterative solutions are the obvious approach to follow, if they can be successfully adapted to free surface problems.